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Aspects of Lorentz violating theories of gravity

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To Lisa
And to my family
For all their love
And for believing in me.

Abstract

Lorentz symmetry is arguably the most fundamental symmetry of physics, at least in its modern conception. On the other hand, some of the issues that plague the currently accepted theory of gravitation could be solved by breaking such symmetry. The theory proposed by Petr Hořava in 2009 brings forward exactly this aspect. The theory, dubbed Hořava gravity, is a UV complete theory of gravity that is also renormalisable. It represents therefore a good candidate for a quantum theory of gravity.

There are some issues though, which typically arise in any theory which explicitely violates Lorentz symmetry. In this thesis we will be concerned with two of these issues, in particular the matter problem and the existence of black holes. The first issue mentioned arises every time we try to couple matter to a Lorentz violating theory of gravity. Indeed, in the matter sector Lorentz symmetry is extremely well constrained, and therefore we need to find a way to avoid the percolation of Lorentz violations to the matter sector through higher order operators. One possible solution based on the separation of scales was proposed in the last few years (Pospelov and Shang, 2012). While studying the proposed mechanism though, the authors uncovered a naturalness problem in the vector sector of the theory. The solutions they proposed relies on the use of some higher derivative terms that are not normally present in the "traditional" Hořava theory. It is unclear then what impact this type of terms can have on the whole theory. In our work we precisely addressed this question. We analysed the perturbations around Minkowski of the most generic theory extended to these type of terms, both from the point of view of the stability of the theory and of the renormalisability. What we found is that the theory retains its renormalisability, but some instabilities occur in the scalar sector. More work is hence required in order to understand whether such instabilities could be tamed, or if the

mixed derivatives should be abandoned in favour of some alternative solution, not presently available.

The second theme we concentrated on is that of the existence of black holes. The definition of black hole in general relativity rely strongly on the causal structure dictated by Lorentz symmetry. As soon as Lorentz symmetry is broken it is therefore unclear whether black holes will still exist. Surprisingly enough black holes have been shown to exist in Lorentz breaking theories, but a rigorous definition was still to be found. In our work we developed the mathematically rigorous definitions for the causal structure of foliated spacetimes and we defined for the first time black holes in such spacetimes. We also uncovered a number of interesting properties of this objects and we developed a local characterisation that allows one to locate horizons without the knowledge of the whole structure of the complete spacetime. Finally we developed the Initial Value Problem for these types of theory in the hope that new simulations of gravitational collapse will be performed using our analysis as a starting point.

♦

The thesis is organised as follows. In the first Chapter we give an introduction on gravity and the problems with its renormalization. We also introduce some of the theories that have been proposed to solve this difficulties. In the second Chapter we start discussing Lorentz violations and we provide a proof of the power-counting renormalizability of a toy model of a Lorentz violationg scalar field theory. We also introduce the theories that we will be studying throughout the thesis. In the third Chapter we discuss the mixed derivative extension to Hořava gravity and we discuss the consequences of the new terms that occur in the theory. In the fourth and fifth Chapters we introduce the causal structure of spacetimes which violate Lorentz symmetry by means of a preferred foliation, we discuss the notion of black holes and horizons and we formalise some results present in the literature adapting them to our framework. In the sixth Chapter we then discuss the Initial Value Problem for such spacetimes, with some attention to the process of gravitational collapse leading to the formation of black holes. Finally in the last Chapter we draw some conclusions and discuss some ideas for future work.

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My PhD, and this very thesis, would not be here today if it weren't for so many people. The number of the people that helped me out and supported me during my course from childhood to what I am now is truly beyon counting, and as much as I wish to thank every single one of them here, a complete list would take the length of a thesis in itself. I will hence refrain from listing all of them, and mention just the most important. Anyone not explicitely mentioned here who nevertheless feels she should be mentioned, don't worry you still deserve my thanks.

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Work is important, and the more the better when it's an enjoyable one. On the other hand, no one can survive the stress of completing a PhD without some channel to relieve said stress. My personal channel in this three years was archery. For this reason I would like to acknowledge here the members of Wilford Bowmen, for their patience, friendliness and for being one of the most welcoming group of people I met. Particular thanks go to Mike T. and Steve, my two coaches, for their teachings and their help all this time. Also, a huge thank goes to Alan, for being a true friend and for the company in numberless training sessions.

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Conventions

In this thesis, we tried to keep the notation and conventions as close as possible to the standard ones followed in the gravitation community. Since this was not always possible (due to the nature of some of the work presented below), in the following we list the conventions and notations which will be adopted throughout this thesis. The hope is that the reader will feel a little less lost in the notation by having a little guide to refer to. In any case, notations will be introduced throughout the thesis at the first occurence, and any possible ambiguity will be stated as clearly as possible.

To start with, the signature of the metric is — unless otherwise stated — taken to be $(-,+,+,+)$ as usually adopted in the gravitational community. Also, units are chosen such that the speed of light $c = 1$ in the infrared. In the cases where it might be more convenient to actually specify the speed of light, we will make that clear.

Also, as for the indices, we will in general adopt the convention used in Wald (1984): when the indices are used to identify the type of tensorial object used, and to contract various tensorial objects, the first letters of the latin alphabet will be used; in gneral it will be enough to use a, b, c, d. When a coordinate system is assumed to be specified, then 4-dimensional indices will be indicated with greek letters – μ, ν, ρ, σ most of the time – and 3-dimensional indices will be indicated with letters taken from the middle of the latin alphabet – usually i, j, k.

In the following we list a series of objects that will be used throughout the whole thesis.

g_{ab} will indicate the general Lorentzian metric; the 3-dimensional metric will be indicated by γ_{ij} when working in ADM coordinates, while it will be indicated by p_{ab} when working in a covariant framework.

In this last case, p_{ab} will also play the role of a projector on the 3-dimensional foliation.

g will indicate the determinant of g_{ab}, γ will indicate the determinant of the ADM 3-dimensional metric while p will indicate the determinant of the covariant 3-dimensional metric.

The ADM variables will be indicated as standard with N for the lapse and N^i for the shift vector.

$\Gamma^a{}_{bc}$ will indicate the connection of g_{ab}.

$\mathcal{R}^a{}_{bcd}$ will indicate the Riemann tensor of g_{ab}.

$\mathcal{R}_{ab}$ will indicate the Ricci tensor of g_{ab}.

$\mathcal{R}$ will indicate the Ricci scalar of g_{ab}.

R_{ab} will indicate the 3-dimensional Ricci tensor (R_{ij} when working in ADM coordinates).

R will indicate the 3-dimensional Ricci scalar.

$\mathcal{T}_{ab}$ will indicate the generic stress-energy tensor.

Publications

The original work reported in this thesis is based on the following published during my doctoral studies:

- Chapter 3: Colombo, Gümrükçüoğlu, and Sotiriou (2015a,b); Coates, Colombo, Gümrükçüoğlu, and Sotiriou (2016);
- Chapter 4 and Chapter 5: Bhattacharyya, Colombo, and Sotiriou (2016b);
- Chapter 6: Bhattacharyya, Coates, Colombo, and Sotiriou (2016a).

CONTENTS

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THE QUANTUM GRAVITY PROBLEM

What is gravity? This is a question that physicists have been trying to answer since the 17th century, with huge successes and blatant defeats, often at the same time.

The "charge" that sources the gravitational field are energy and momentum. For this reason as soon as matter is present, gravity will be as well. At the same time gravity is a long range force; what this means is that a test particle will be able to "feel" the gravitational attraction of a mass distribution even when the distance between the two increases to infinity. The combination of these two factors made gravity the first interaction to be recognised and studied. However, the hierarchy problem arising from the fact that gravity is much weaker than all the other interactions, makes it very difficult to test. Further problems are encountered when we try to make gravity and quantum mechanics play together. To date there are many proposals for a quantum theory of gravity, none of which is universally accepted as the solution.

This thesis is a reflection on this problem. Naturally we have no pretence to have solved the problem. Rather, this is an attempt to develop further one of the most controversial solutions which recently joined the scene of "perhaps quantum gravity theories": the possibility of solving the problem by breaking Lorentz invariance.

Throughout this Chapter we will give a brief introduction about gravity theories, how they developed and what are the mainstream proposals for a quantum theory of gravity. We will then proceed in the next chapters to introduce the theories we chose to reflect upon and the results obtained during my PhD.

1.1 A BRIEF HISTORY OF GRAVITATION

The first fully fledged physical theory to include gravitation, Newtonian theory, was laid out in 1687 in Sir Isaac Newton's "Philosophiæ Naturalis Principia Mathematica" (Newton, 1687). This theory was the most complete description of nature available for quite a long time and indeed it managed to explain a good deal of the physical phenomenology known at the time, from the mechanics of moving objects on the Earth, to the laws of gravitation and the dynamics of the Universe. In fact this last aspect can be probably considered one of the most important contributions to physics of all times. Newton's law of gravitation managed to explain, among others, the motion of planets around the Sun with incredible precision. It was also able to describe the dynamics of many other celestial bodies even outside the solar system, such as comets. At the same time, the laws discovered by Newton describe with incredible precision the dynamics of macroscopic bodies on the Earth.

Thanks to the precision of its predictions, this theory was considered the final physical theory for many years after its discovery. Research in physics therefore concentrated mainly on developing every possible aspect of Newton's theory, and on applying it to a wider and wider range of phenomena.

It was only at the end of the 19th century that technological development allowed for the discovery of more and more curious phenomena, which were not necessarily possible to explain using Newtonian physics. These new phenomena were identified on two opposite and very different length scales: atomic physics at very small scales, and gravitational physics at very large ones. As an example, among the new phenomena at very small scales we can think of the Rutherford experiment and the discovery of the photoelectric effect. On the other hand, at very large scales, some observations at solar system level — first and foremost the measurement of the precession of the perihelion of Mercury — were found to be in disagreement with the theory of gravitation proposed centuries before by Newton.

This novel understanding of nature led a great deal of people to start rethinking physics, and some game-changing discoveries were made in the following few years. The fascinating fact about these discoveries is that they revolutionised forever the understanding of physics, both from a philosophical and a technical point of view.

The first great philosophical and technical revolution was due to Albert Einstein. In the theory which came to be known as *special relativity* (Einstein, 1905), he realised that space and time — conceived as absolute and not related to each other in Newton's theory — are actually cut out of the same cloth. Instead of having space and time as separate and independent entities, we have one single entity which we call *spacetime*. The fundamental symmetry attributed to physical systems in a vacuum, namely rotational symmetry, extends to more general spacetime rotations called *Lorentz transformations*. One consequence — or one of the assumptions necessary to derive Lorentz transformations, depending on which way we choose to approach the problem — is that the speed of light c is constant for any observer, and no signal can carry information at a speed higher than c.

Extending this idea further, Einstein managed in the following years to incorporate gravity in his theory, thus giving birth to *general relativity* (Einstein, 1915, 1916). The main idea in this case is that, instead of having some unaccounted for "gravitational force" acting on particles, said particles simply follow the "straightest" lines in the spacetime. At the same time, the distribution of matter in the spacetime deforms spacetime itself according to Einstein equations; as a consequence what "straight" lines, or *geodesics*, look like will depend on the shape of the spacetime. The simplest way to see this is to think that spacetime dictates to matter how to move, and matter dictates to spacetime how to shape. This idea is simple but extremely deep at the same time, and indeed produced a revolution in how we conceive gravity itself.

One last important aspect of this theory is that, at sufficiently small scales, spacetime is to a good approximation flat. Lorentz symmetry will hence hold locally.

As a confirmation of this picture, experimental verification of the predictions of general relativity arrived almost immediately after its introduction. The most impressive and important verification was the observation that light rays curve under the influence of the gravitational field, exactly as predicted by Einstein; the direct observation was made by Sir Arthur Eddington during a solar eclipse, and is reported in Dyson et al. (1920). The second early confirmation of the validity of GR came with the calculation of the precession of the perihelion of Mercury, found to be perfectly in line with what was observed. More observations followed in the years, all essentially confirming the validity and predictive power of GR. Notably, a very recent technical breakthrough was the direct measurement of the gravitational waves coming from a black hole-black hole merger (LIGO Collaboration, 2016a). In addition to confirming the existence of gravitational waves and black holes, both predicted by GR, tests of general relativity using the data collected have been performed and no statistically significant deviation from its predictions have been found (LIGO Collaboration, 2016c).

1.2 GRAVITY IN A QUANTUM WORLD

The development of a new understanding of gravity, surpassing both from the conceptual and computational point of view the theory laid down by Newton, can rightfully be considered one of the great discoveries of the 20th century, and helped to shed light on the new interesting phenomena that were observed at large scales. On the other hand, GR didn't contribute to undertanding the wealth of new and puzzling results deriving from experiments in atomic physics at very small scales. Two decades of research and experiments thus led to the second major discovery of the 20th century: the world is described by *quantum mechanics*. At large enough scales quantum effects may well be ignored, but at small scales they play a paramount role that cannot be neglected.

The discovery of quantum mechanics changed at a fundamental level the way we understand physics, first and foremost on a philosophical level. Before this discovery, in both Newtonian theory and in Einstein's relativity theories, particles were considered as points moving in space, or spacetime, occupying a precise position with a determined velocity at each point. Quantum mechanics overthrew this understanding and introduced uncertainty as a fundamental concept in a theory for the first time in the history of science. Particles in quantum mechanics — even slow moving ones — are described by a probability distribution in space, not by a determined position (see Dirac (1930) for an early but nonetheless complete introduction). Also, some quantities that were considered as continuous in Newtonian physics were more naturally thought of as discretised. The most important example of this fact is the energy of a particle in a bound state: in quantum theory energy cannot take any value, but rather comes in integer multiples of "quanta" of fixed energy — hence the name quantum mechanics.

The most important consequence of these discoveries is that we cannot predict with certainty the development of a physical system, and an outcome can be computed only with a certain probability. This is unlike the case of Newtonian mechanics where the development of the entire Universe is predetermined by the knowledge of the initial conditions for every particle and could — at least in principle — be computed with absolute precision. The importance of quantum mechanics is not just philosophical though. It also introduced new methods for computation, thus contributing to the development of the mathematical instruments needed to deal with the calculations, and its findings have more and more applications in the technological scene.

After the discovery of quantum mechanics, which concentrated on the quantum description of particles, and after the numerous experimental confirmations of this new — and for many aspects counterintuitive — understanding of nature, the natural extension was the attempt to extend the already existing classical field theory to a quantum theory of fields. This started the long study of what is now known as *Quantum Field Theory*; the study of this theory is still underway and provides us with a number of challanges even today. The first examples of a quantum field theory are the scalar field theory proposed by Klein and Gordon and *Quantum Electrodynamics*, the quantum theory of the electromagnetic field. Since fields are naturally relativistic, as the propagation of signals can happen at the speed of light, QFT represents the special relativistic version of quantum mechanics. This approach after a first period of difficulty — the reasons for which are described in the next section — gathered exceptional success. For example, it allowed people to compute for the first time the fine-structure constant, i.e. the fundamental constant of electromagnetism, with incredible precision and later to unify three of the four fundamental interactions, namely the Electromagnetic, Weak and Strong interactions, into what is nowadays known as the Standard Model of particles.

The next step that comes naturally to mind is to unify general relativity to quantum theory, in order to take in account gravitational effects in a theory which until now has been describing a quantum world without gravity. This is not as easy as one could think, for reasons that will become apparent in the next Section, and attempts at a complete theory have not been successful so far.

One of the tenets of modern physics is that quantum theory is the correct description of nature; the search for a *quantum theory of gravity* is therefore of great importance and has been topical for the last few decades. To date, the search for such a theory is still under way, with many possible proposals for a solution, and yet no definite solution at hand. In the remainder of this Chapter, we will discuss the major difficulties encountered when attempting a quantum description of gravity, and we will briefly describe some well-known attempts at a solution that have been proposed in the last years.

1.2.1 *The problem with renormalisability*

A generic property of quantum field theories is that, in the calculation of the scattering amplitude associated to some Feynman diagram, momentum integrals will in some cases "blow up" as the energy increases. The infinities coming from such diverging integrals are not physical, but nonetheless they need to be taken care of in order to have a theory capable of delivering sensible predictions. This was first noticed within quantum electrodynamics, and a great deal of effort went into understanding where the infinities come from and how to get rid of them.

There are two aspects to the process of dealing with such infinities. The first is normally referred to as *regularisation* and consists — at least loosely speaking — of adding a cutoff to the theory. This cutoff represents the (energy) scale above which the theory can no longer be trusted, mainly due to new physics that is expected to start playing a role, and will effectively limit the portion of phase space over which the integrals are evaluated.

On top of regularising the theory, often the need arises to perform what is usually called *renormalisation*. In a nutshell, this technique consists of adding to the perturbative expansion of the theory some terms that allow, by modifying the couplings, to "subtract away" the divergences and therefore to obtain a finite result. The intuitive justification of such a procedure is the understanding that what we measure in an experiment is really the difference between two quantities, and not one quantity directly. This way, by renormalising the theory, we implicitely rewrite the theory itself in a more physical form, allowing thus to find more sensible predictions. As mentioned, this technique can be used perturbatively and therefore doesn't necessarily cure the complete theory, but rather the truncation we chose to consider. In fact, a theory is considered renormalisable if it is renormalisable at every order of perturbation.

In the last few decades a great host of renormalisation techniques have been developed; we won't go in any more detail here as this is not the primary concern of this thesis, but many sources are available (see as an example Peskin and Schroeder, 1995) with detailed discussions regarding this aspect.

It is important to note though that a theory does not necessarily have to be renormalisable. Indeed, it could happen that in the renormalisation process we are required to add an increasing number of terms — called counterterms — while working through the perturbation orders, to be able to cancel out all the divergences we encounter. If this happens, then the number of counterterms will eventually diverge, and the theory is said to be *non-renormalisable*. For this reason, a major element to consider in this process is to be able to assess when a theory can be renormalised, possibly without having to embark in the whole process in vein. The easiest way to do so is called *power counting*; the idea is that by performing the dimensional analysis of the lagrangian of the theory we have at hand, we could be able to determine whether the theory can be renormalised. We will be using this technique later on, so we will avoid giving a detailed account here. We need to note though that this method is not completely reliable, and in some cases a theory that seems to be non-renormalisable at a power counting level could still be renormalisable using more sofisticated techniques.

After this rather lengthy but nonetheless necessary introduction, we can finally understand the problems arising when we consider gravity in a quantum world. The issue is precisely as just explained: if we accept general relativity as the correct theory of gravity, then gravity cannot be renormalised when used as a quantum field theory. To show this we will use a very simple argument, which can be found in Zee (2010, Part III). Let us consider an hypothetical graviton-graviton scattering. The coupling constant in this process will be Newton's constant G_N. The mass dimension of this coupling is -2 and therefore the theory is automatically non-renormalisable, already at a power counting level.[1]

A further problem is encountered (see Zee, 2010, Part VIII) when we try to write general relativity as a quantum theory. To do so, let's consider the Einstein-Hilbert action

$$\mathcal{S}_{\text{EH}} = \frac{1}{16\pi G_N} \int \mathrm{d}^4x \, \sqrt{-g}\mathcal{R} \,, \tag{1.1}$$

and rewrite the action for perturbations around flat space $\mathrm{g}_{ab} = \eta_{ab} + h_{ab}$, where η is the Minkowski metric and h is the perturbation. The action (1.1) can be then written schematically as[2]

$$\mathcal{S}_{\text{EH}} = \frac{1}{16\pi G_N} \int \mathrm{d}^4x \, \Big(\partial h \partial h + h \partial h \partial h + h^2 \partial h \partial h + \dots \Big) \ . \tag{1.2}$$

Due to the presence in (1.1) of $\sqrt{-\mathrm{g}}$ and of the inverse of the metric, this series expansion is infinite. This has an impact on the renormalisability of the theory, since — on top of the obvious problem with having a theory expressed by an infinite series of terms — this implies that there is an infinite number of self-interactions between gravitons, and all these self-interactions will produce divergent momentum integrals. It is however natural to think that gravitons will self-interact: after all they are fields and any field in general relativity will interact with the gravitational field; the gravitational field though is mediated precisely by gravitons, and therefore they cannot do anything but interact with themselves. Despite self-interactions not being per se a problem for the renormalisability of a theory, the fact that we can have an infinite number creates the conundrum of a non-renormalisable theory.

The fact that the currently accepted theory of gravity is not trivially renormalisable when conceived in a quantum field theoretical setting is a problem. This is the reason why researchers in theoretical physics, in the last few decades, concentrated most of their efforts towards understanding how

1 Again, as noted above, this doesn't preclude the possibility to be able to renormalise the theory using more sofisticated techniques.

2 Notice that we dropped the tensorial indices to reduce the clutter in the expression.

to unify gravity and quantum theory. Such an endeavour might seem trivial and unimportant, but is actually fundamental for the advancement of knowledge in the field. Indeed, without a quantum theory of gravity we arguably cannot fully understand many aspects of gravitation that have been uncovered throughout the years. Among the possible examples we can mention what happens near to spacetime singularities — both the ones that lie at the center of black holes and the one at the beginning of the Universe — or other aspects of black hole physics that exhibit quantum effects and that are not explained in a satisfactory way by the currently available theories. In this respect, one may think first and foremost about black hole thermodynamics (Bekenstein, 1973) and the Hawking effect (Hawking, 1974, 1975), with all the paradoxes they carry along.

1.3 possible solutions

In the last few decades, a number of theories have been proposed with the broad goal of overcoming the difficulties encountered when unifying gravity and quantum theory. We don't have the space to discuss all the diverse variety of possible theories that were proposed, but it might be interesting to introduce at least the major players in the quantum gravity scene.

1.3.1 *String theory and M-theory*

String theory was born at the beginning of the 70's as a theory to model the strong interaction (for a simple and up-to-date introduction see Conlon, 2016). After the discovery by Veneziano of the peculiar form of the scattering amplitude for strong interacting mesons (Veneziano, 1968), it was realized that such amplitudes could be derived by a model of the vibration of a relativistic string. The underlying idea that started to bloom was that instead of particles, the fundamental entities could be strings, either closed or open, and that the particle picture would be recovered in the limit where the typical length scale is much bigger than the size of the string. This interpretation also had some very peculiar characteristics, notably the fact that it was shown to be consistent only in a 26-dimensional space.

This model was later shown to not be very precise in explaining strong interactions, as it only worked in some approximation, and therefore quantum chromodynamics was introduced with much more success. After this, string theory laid dormant for a few years, just to come back to the scenes with the advent of supersymmetry. The discovery of supersymmetry led people to realise that a supersymmetric version of string theory existed, it needed "only" ten dimension to be consistent and one of the possible string states did describe the graviton. This way, string theory was abandoned as a theory of the strong interaction, just to be rediscovered as a unified theory possibly capable of describing all the fundamental interactions including gravity.

Various theories that behaved in similar ways were discovered, but new understanding brought the unification of all these different string theories into a single framework, called *M-theory*. The various string theories known at that point were just different sectors of such bigger framework. For this reason it is more correct to think of M-theory as framework rather than as a theory, in that it encompasses a number of different "sub-theories" capable of describing different aspects of nature.

The legacy of string theory is quite an important one. It might in the first place help in solving some of the problems faced when unifying gravity and quantum field theory, and allows one to perform calculations in a wide range of fields. Additionally many advances in other fields — primariliy in mathematics — were triggered by string theory, in the search for new technologies to be able to solve the complicated calculations people were faced with.

The latest breackthrough in this field came along with the discovery of the so called *AdS/CFT duality*, proposed by Juan Maldacena (Maldacena, 1999). In a nutshell, the original incarnation of this duality relates a realisation of string theory (or M-theory) in 5-dimensional Anti-de Sitter space to a Conformal Field Theory on the boundary of the space. Further work has been done in this direction in the following years and now this duality encompasses a much more general landscape of possible relation between different theories. Without going into any detail of how this works, the great importance of this discovery is that it allows one to relate a gravitational theory in the bulk to a quantum field theory on the boundary. As this field theory lives on the boundary of the space, gravity is not a problem anymore because the boundary of Anti de Sitter resembles locally Minkowski spacetime, and hence gravity is not present. This way, once the dictionary to go from one setting to the other has been developed, we can compute quantities in whichever theory it is easier to do so, and then infer information in the other theory. This discovery brought along a renewed interest in M-theory and soon a great deal of applications came, ranging from condensed matter to black hole physics — notably the calculation of the Hawking temperature in this setting — and quantum gravity.

As we see, M-theory does indeed represent a possible good candidate to solve the problem at hand. On the other hand this framework hinges on a number of assumptions which were never shown to actually hold, first and foremost the existence of supersymmetry and extra dimensions, and therefore its validity is still debated. The community working on this framework is probably the biggest in the field of quantum gravity, but other possibilities are indeed present.

1.3.2 *Quantum spacetimes and background independent theories*

Despite being a revolutionary theory in many aspects, string theory still considers spacetime as a continuum, an idea which is almost pre-relativistic.

Indeed, with Einstein's discovery of general relativity, spacetime in a sense disappeared by being replaced by a field, namely the gravitational field (Rovelli, 2003). For this reason, towards the end of the 80's, work by a number of people led to a theory where spacetime itself is quantised. In the same way that photons are taken as the excitations of the electromagnetic field, spacetime is considered to be made up by loops which represent the excitations of the gravitational field. This set of loops builds a network, dubbed a *spin network* by Roger Penrose (Penrose, 1971). This picture only describes space, and hence we need a well defined picture of time evolution. This came along shortly after by defining the history of the spin network as a *spin foam*, which represents the time evolution of the spin network. This idea gained ground throughout the years, becoming eventually what is known nowadays as *loop quantum gravity*; this theory is another major player in the search for a quantum theory of gravity. The theory also brought a good deal of successes, first and foremost for being able to compute — up to a factor called Immirzi parameter — the dependence of the black hole temperature on the area of the event horizon. As a last remark it is clear that, since a background spacetime does not exist in this theory but is actually created together with its dynamics, the theory is formally background independent.

There are on the market also other possibilities for background independent theories which consider the spacetime as quantised. The best known of such theories is the so called *Quantum Gravity from Causal Dynamical Triangulation* (CDT for short) introduced for the first time in Ambjorn and Loll (1998) — see also Ambjorn et al. (2010) for a more recent review. The idea underlying this approach is to try to define a quantum gravity relying on nothing but standard principles from QFT and on ingredients and symmetries already contained in general relativity. CDT is hence a non-perturbative implementation of the gravitational path integral, or a "sum over histories," using a triangulated spacetime as a field regulator (Ambjorn et al., 2010). This approach has claimed some successes since its implementation, as it features a continuum limit which seems to produce the correct classical limit. On top of that, the ability to carry out numerical simulations in a relatively easy way is an added bonus, in that it allows one to test the predictions of the theory and compare them with our current understanding of the Universe. A notable example of this is the good agreement between some features of the emerging spacetime found in CDT and the spacetimes that are used to describe our Universe at late times.

A final possibility for a theory with a quantum spacetime is given by the so called *causal sets* theory. This theory, proposed originally in Malament (1977), consists in considering the spacetime as discrete while retaining Lorentz Invariance by construction. The geometric information is contained in the causal relations between points and thus the metric of a globally hyperbolic spacetime can be reconstructed up to a conformal factor. Using this technique the quantum dynamics of a spacetime can be obtained. This theory is currently being studied in more and more detail, so we won't delve more in this since it's outside the scope of this thesis. A more detailed review can be found in Wallden (2010).

1.3.3 *Further possibilities*

Along with the proposals listed above, which are the ones that see the biggest communities, there are some more options being explored.

The first option is to employ non-perturbative techniques to renormalise a quantum theory of gravity. This idea was first introduced in Weinberg (1979) and relies on the existence of a fixed point in the renormalisation group flow for gravitational couplings. This idea had a number of successes in the past, but is still being investigated in relation to a number of problems. For a more detailed review see Percacci (2007).

A further candidate proposed in the 1980's, the so called $N = 8$ Supergravity is the most modern incarnation of the application of supersymmetry to a gravitational theory. This proposal was abandoned in favour of string theory in the past, but the interest of the community is being renewed recently.

One last approach at the unification of gravity with the remaining forces has been carried out recently, and hinges on rederiving the known physical theories while assuming the geometry of spacetime to be non-commutative (for a review on this approach see Van den Dungen and van Suijlekom, 2012). While promising, this idea is still a work in progress, the principal difficulties being due to the lack of a complete and rigorous mathematical theory of quantum fields.

♦

The main difficulty with all these proposals for a quantum theory of gravity is linked to the inability of all the theories we have available at the moment to produce concrete predictions that can be tested within the available experimental observations. For this reason, with no experimental lighthouse to guide us in the right direction, all the proposals depicted above, despite their fundamental differences, could or could not be considered the right solution. In addition, none of the proposals available at the moment describes perfectly all the phenomenology known to us.

It is not currently possible to determine which, if any, could be the correct solution. It is therefore a very personal "choice", and people will be led to "believe" in any of the possibilities depending on which they feel delivers better prospects.

2

LORENTZ BREAKING TO THE RESCUE

Lorentz symmetry is considered the fundamental symmetry of modern physics. In fact, Lorentz symmetry is the fundamental symmetry of special relativity, and by extension of quantum field theory — at least when considered in a flat spacetime — and also a local symmetry of general relativity, thanks to the local flatness theorem. On the other hand, there is no experimental confirmation of this symmetry being a fundamental symmetry of nature, instead of an effective symmetry that emerges at low energy.

In addition, the various proposals for a quantum theory of gravity that we discussed before present us with the possibility of Lorentz violations as part of the theory. This aspect is debated, but in most of the proposals Lorentz symmetry has never been shown to be fundamental; notable exceptions to this are asymptotic safety and causal sets, where as we discussed before Lorentz symmetry is present by construction. In fact, in recent years there have been claims that in both String (M) theory (see e.g. Kosteleky and Samuel, 1989; Mavromatos, 2007) and LQG (see e.g. Gambini and Pullin, 1999) there could be violations of Lorentz symmetry.

Given this premises, it is apparent that we have a compelling duty to test Lorentz violations in nature. The reason for this is twofold: in the first place, being this symmetry so important in modern physics, we have the moral obligation to test it as thoroughly as we can. In addition, since violations of this symmetry might be already present in candidate theories of quantum gravity, it would be interesting to understand if this can actually be a viable assumption or if Lorentz violating quantum gravity has to be ruled out on phenomenological grounds. It is interesting to note — and the importance of this fact will be discussed extensively in the following Chapter — that Lorentz symmetry is constrained in an extremely precise way in the matter sector, chiefly from particle physics experiments (see for a review Kosteleky and Russell, 2011). In the gravitational sector such constraints are instead much weaker, and therefore it could be interesting to try and test Lorentz violations by employing astrophysical and cosmological tests. Recent discussions on this aspect can be found in Mattingly (2005) and Liberati (2013).

There is a further reason why Lorentz violations might play an important role in physics, and this reason is the one we are most interested in for the present work. It can be shown — and the next Section will be devoted precisely to a discussion of this particular aspect — that Lorentz violations can have beneficial effects on the renormalisability of QFTs. It might be worth noting here that adding higher curvature terms to the gravitational action also helps with renormalisability. However, as was pointed out in Stelle (1977), doing so introduces ghosts in the theory. These two facts are the starting point towards the formulation of Hořava gravity, as will be discussed in more detail later.

Given the considerations above, a legitimate question when it comes to gravity is whether there could be an advantage in considering Lorentz symmetry as an emergent, approximate symmetry at low energies, instead of assuming it to be a fundamental symmetry. Indeed, what we saw in the previous Chapter is that the main problem we are presented with when trying to model general relativity as a quantum theory is the fact that the resulting quantum field theory is not renormalisable. In fact it has been shown (Hořava, 2009a,b) that by allowing for Lorentz violations the renormalisability of the theory improves, producing therefore a strong candidate for a quantum gravity theory.

Proposals for Lorentz violating theories of gravity have been put forward in recent years: our main target for the remaining of this thesis will be to consider them in detail. For the remainder of this Chapter, we will show how violating Lorentz symmetry can help with the renormalisability of field theories, and we will discuss the theories that have been proposed as Lorentz violating gravity candidates.

2.1 GENERALITIES AND THE LIFSHITZ SCALAR

We mentioned above how volating Lorentz invariance could lead to some advantage when it comes to renormalisability of field theories. This idea can then be applied to general relativity — when considered as a field theory — and perhaps this will bring some advantage. Before starting to discuss the case of gravity though, it is convenient to start with a simpler and in some way more general approach, in order to show how the advantages to renormalisability come about in a generic theory. This will also help in developing the techniques we will be using — or at least refer to — later on.

The simple case we wish to consider is the *Lifshitz scalar theory*. In a nutshell, we will consider a self-interacting scalar field theory with anisotropic scaling; this last aspect means that the dimensional scaling of spatial coordinates is different than that of the time coordinate. The reason for considering such theory is that it violates Lorentz symmetry in a similar fashion to the gravity theories we wish to consider later on. This will allow us to transport the intuition coming from such theory to the case of gravitational theories. It has to be pointed out that the best arguments available to date for the renormalisability of Lorentz violating gravity theories comes from the analogy with the Lifshitz scalar that we will discuss below. Such arguments were originally introduced in Visser (2009a,b).

2.1.1 *Renormalising the Lifshitz scalar*

Let's consider a scalar field theory in flat $(1+D)$ – dimensional spacetime, described by an action given as

$$\mathcal{S}_{\text{free}} = \int \mathrm{d}t\, \mathrm{d}^D x \, \left[\dot{\phi}^2 - \phi(-\triangle)^z \phi\right] \, , \tag{2.1}$$

where $\triangle$ is the spatial Laplacian and z is some integer number. Since both terms in the action should have the same dimension we see easily that we need to impose the scaling of the spacetime coordinates to be *anisotropic*; such scaling will then given by

$$t \to b^{-m} t \, , \qquad x^i \to b^{-1} x^i \, . \tag{2.2}$$

Additionally, unless we are willing to introduce a dimensional parameter in front of one of the terms of (2.1), the easiest way to have the same dimension is to impose the scaling dimension to be the same as the number of spatial derivatives. We will hence assume – at least for the moment – that $m = z$.

The dimensions of spacetime quantities are given by

$$[\mathrm{d}t\,] = [\mathrm{d}x\,]^z \, , \qquad [\partial_t] = [\boldsymbol{\nabla}]^z \tag{2.3}$$

and since we require the action to be dimensionless, the dimension of the scalar field can be easily computed from the first term to be

$$[\phi] = [\mathrm{d}x\,]^{(z-D)/2} \, . \tag{2.4}$$

We can introduce at this point formal symbols κ and ω, having respectively the dimension of momentum and energy, such that

$$[\kappa] = [\mathrm{d}x\,]^{-1} \, , \qquad [\omega] = [\mathrm{d}t\,]^{-1} \quad \Rightarrow \quad [\omega] = [\kappa]^z \tag{2.5}$$

and we will find for the scalar field

$$[\phi] = [\kappa]^{(D-z)/2} = [\omega]^{(D-z)/2z} \, . \tag{2.6}$$

Notice that if we choose isotropic scaling $z = 1$, we recover the standard results well known from scalar field theories in QFT.

As a side remark, notice that the choice of units we made prevents us from setting $[c] = 1$ as we usually do. The reason for this is easily understood by considering the arguments given until now. Indeed we can see directly from the dimensional analysis that

$$[c] = [\mathrm{d}x \,/\mathrm{d}t\,] = [\kappa]^{z-1} \; ; \tag{2.7}$$

again for $z = 1$ the usual result is recovered where c is dimensionless[1].

At this point, let's add some generic self-interactions to the theory, in the polynomial form given by

$$\mathcal{S}_{\text{int}} = \int \mathrm{d}t \, \mathrm{d}^D x \, \sum_{n=1}^{N} g_n \phi^n \; . \tag{2.8}$$

Employing the results given above, we can compute easily the dimension of the couplings to be

$$[g_n] = [\kappa]^{D+z-n(D-z)/2} \; ; \tag{2.9}$$

the coupling therefore has non-negative momentum dimension provided

$$D + z - \frac{n(D-z)}{2} \geqslant 0 \; . \tag{2.10}$$

Recall a classic result derived in QFT which tells us that non-negative dimension of the couplings indicates that a theory is power-counting renormalisable. Since D, z and n are all positive by definition, we see that (2.10) is solved by either imposing

$$n \leqslant \frac{2(D+z)}{D-z} \qquad \text{for } z < D \tag{2.11}$$

or by imposing

$$n \leqslant \infty \qquad \text{for } z \geqslant D \; . \tag{2.12}$$

We have therefore already strong indications that the theory is power-counting renormalisable, provided z is at least as large as D.

1 Choosing units where $[c] = 1$ is of course a possibility; it has to be noted though that, while the physical content of the arguments given in this Section will not change, the treatment can become quite more cumbersome. We choose therefore not to consider this possibility and accept instead $[c] \neq 1$.

In order to show in a more rigorous way that the theory is power-counting renormalisable, we will now calculate the superficial degree of divergence for a generic Feynman diagram in this theory.

2.1.2 *Superficial degree of divergence*

Consider a generic Feynman diagram[2]. For each loop in the diagram we pick up an integral in the form

$$\int \mathrm{d}\omega\, \mathrm{d}^D k\, [\cdots] \,. \tag{2.13}$$

Any line integral will also be characterised by a propagator of the form

$$G(\omega, \boldsymbol{k}) \propto \frac{1}{\omega^2 - [m^2 + \cdots + k^{2z}]} \,. \tag{2.14}$$

We are, as before, interested in the dimensional counting of this elements; we can then see that the momentum space volume element has dimensions

$$\int \mathrm{d}\omega\, \mathrm{d}^D k \quad \rightarrow \quad [\mathrm{d}\omega\,][\mathrm{d}k\,]^D = [\kappa]^{D+z} \,, \tag{2.15}$$

while propagators have dimension

$$G(\omega, \boldsymbol{k}) \quad \rightarrow \quad [\kappa]^{-2z} \,. \tag{2.16}$$

Consider now a generic Feynman diagram F, with L loops and I internal lines: the dimension of the whole diagram will be

$$[F] = [\kappa]^{(D+z)L-2Iz} \,; \tag{2.17}$$

the superficial degree of divergence is therefore by definition

$$\delta = (D+z)L - 2Iz \,. \tag{2.18}$$

2 If we were trying to show that power-counting renormalisability is not satisfied, it would be enough to consider only some particular diagram which proves the point. Since what we will aim to show instead is that the theory is power-counting renormalisable, we need to consider a generic diagram.

This last result can be rewritten in a more convenient way as

$$\delta = (D - z)L - 2(I - L)z \; , \tag{2.19}$$

and since we need at the very least L propagators to be able to create L loops (i.e. $I \geqslant L$) we then see easily that

$$\delta \leqslant (D - z)L \; . \tag{2.20}$$

We find then that, provided $z \geqslant D$, we will have

$$\delta \leqslant 0 \; ; \tag{2.21}$$

this means that the worst possible divergences we can encounter are logarithmic, and pose no threat to renormalisability. In addition $\delta = 0$ can only arise for $L = I$, which produces to the so called "rosette" diagram; this type of diagram can be eliminated immediately by normal ordering.

As a last thing, we would like to consider the case of derivative interactions. This type of interactions are not usually considered in a scalar field theory, but in our case the whole analysis is done with the final goal of applying the intuition gained to a gravity theory. As we discussed previously, in a gravity theory self interactions of gravitons are inevitable, and therefore it would be interesting to check what happens — at least in an intuitive way — when such interactions are taken in account. To do so we will use some additional interaction term (Visser, 2009b) given by

$$\mathcal{S}_{\text{int}} = \int \mathrm{d}t \, \mathrm{d}^D x \, P(\nabla^{2z}, \phi) \; , \tag{2.22}$$

where $P(\nabla^{2z}, \phi)$ is an infinite order polynomial containing at most $2z$ spatial derivative. In this case, the vertices of an arbitrary Feynman diagram will carry up to $2z$ factors of momentum. Additionally, only internal momenta will count towards the superficial degree of divergence and therefore δ will be given by

$$\delta \leqslant (D + z)L + 2z(V - I) = (D - z)L + 2z(V + L - I) \; . \tag{2.23}$$

Making use of the well known identity for graphs $V + L - I = 1$, we then have that the superficial degree of divergence becomes

$$\delta \leqslant (D - z)L + 2z \ , \tag{2.24}$$

and therefore for $z \geqslant D$ we simply obtain

$$\delta \leqslant 2z \ . \tag{2.25}$$

We can see from this last expression that the superficial degree of divergence is bounded from above by the canonical dimension of the operators already explicitely included in the bare action; this is a standard sign of power-counting renormalisability (Visser, 2009b).

Relying on the arguments presented in this Section, we can then claim that Lorentz violating scalar theories with anisotropic scaling, including ones which feature derivative interactions, can be made to be power-counting renormalisable. This does not prove that the theory is renormalisable, since there could still be some divergences hidden inside subdiagrams, but it is at least a strong indication that it can be and it provides a good starting point towards discussing in more depth the renormalisability properties of the theory.

2.2 LORENTZ VIOLATING GRAVITY THEORIES

We have seen in Section 2.1 how Lorentz violations can be a blessing in disguise when it comes to the renormalisability of field theories. This fact led some to realise that this could be one possible way to deal with the problems in renormalising gravity as a quantum field theory. Lorentz symmetry, on the other hand, has been a pillar for physics since the last century and hence we need to understand if a theory of gravity without Lorentz symmetry is viable in the first place.

In the following we will explore two of the possibilities that have been proposed in the last few decades as Lorentz violating gravity theories. We will start with exploring Einstein-Æther theory, introduced as a generic theory to study the properties of Lorentz violating gravity, and we will then give an overview of Hořava-Lifshitz gravity, a fully fledged proposal for a renormalisable quantum theory of gravity.

2.2.1 *Einstein-Æther theory*

Einstein-Æther theory is one of the most well-studied among the Lorentz violating gravity theories available. It consists of general relativity coupled to a vector field, the *æther*, that is everywhere timelike and never vanishes. The role of such vector field is simply to dynamically introduce a preferred frame and thus break Lorentz symmetry.

Einstein-Æther theory was first introduced in Jacobson and Mattingly (2001) as a tool to test the effects of violations of Lorentz symmetry in a gravity theory; for a more recent review see Jacobson (2007).

The action is the usual Einstein-Hilbert action plus a part involving the æther. Since — at least for the time being — we are interested in studying the effects of Lorentz violations in the gravity sector, matter is only coupled to the metric in the same fashion as in GR. For this reason we disregard for now the matter action; the action of vacuum Einstein-Æther theory is then given by (Eling and Jacobson, 2004; Jacobson, 2007)

$$\mathcal{S}_{\text{Æ}} = \frac{1}{16\pi G_{\text{æ}}} \int \mathrm{d}^4x \sqrt{-\mathrm{g}} \left[\mathcal{R} + \mathcal{L}_u\right] \,, \tag{2.26}$$

where $\mathcal{R}$ is the Ricci scalar and the æther lagrangian is given by

$$\mathcal{L}_u = -Z^{abcd}\nabla_a u_c \nabla_b u_d + \lambda \left(\mathrm{g}^{ab}u_a u_b + 1\right) \,. \tag{2.27}$$

In the above expressions, the tensor Z^{abcd} is given by

$$Z^{abcd} = c_1 \mathrm{g}^{ab}\mathrm{g}^{cd} + c_2 \mathrm{g}^{ac}\mathrm{g}^{bd} + c_3 \mathrm{g}^{ad}\mathrm{g}^{bc} - c_4 u^a u^b \mathrm{g}^{cd} \,. \tag{2.28}$$

λ represents a Lagrange multiplier used to guarantee the æther vector to have unit norm. The variation of the action with respect to λ produces

$$u_a u^a = u^2 = -1 \,, \tag{2.29}$$

and this way the æther is forced to be unit timelike everywhere in spacetime.

The field equations and the equations of motion for the æther are obtained by varying the action of the theory with respect to the metric and the æther vector.

Variation with respect to the æther gives the vector equation of motion

$$\nabla_b J^b{}_a + c_4 a_b \nabla_a u^b + \lambda u_a = 0 \tag{2.30}$$

where we have defined the acceleration of the æther as $a_b = u^a \nabla_a u_c$ and $J^a{}_c = Z^{ab}{}_{cd} \nabla_b u^d$.

Variation with respect to the metric produces the field equations

$$\mathcal{G}_{ab} = \mathcal{T}_{ab} \ , \tag{2.31}$$

where the stress-energy tensor for the æther is given by

$$\begin{aligned} \mathcal{T}_{ab} &= \nabla_c \left(J_c{}^{(a} u_{b)} + J_{(ab)} uc + J_{(a}{}^c u_{b)} \right) + c_1 \left(\nabla_a u_c \nabla_b u^c - \nabla^c u_a \nabla_c u_b \right) \\ &\quad + c_4 a_a a_b + \lambda u_a u_b + \frac{1}{2} g_{ab} \mathcal{L}_u \ . \end{aligned} \tag{2.32}$$

Linearising the theory around flat spacetime and analysing the propagating degrees of freedom we find that, since the action of the theory contains also the æther vector, there are more propagating modes than in standard general relativity. In particular there are two transverse traceless tensor modes (spin-2), representing the usual gravitons of GR, two transverse vector modes (spin-1) coming from the transverse part of the æther vector and finally a longitudinal scalar mode (spin-0). All this modes are massless and have different constant speeds of propagation, given as combinations of the c_i couplings (see e.g. Jacobson and Mattingly (2004); Jacobson (2007) for the explicit expressions).

As a last remark, we note that the values of the couplings c_i can be constrained with the use of some consistency requirements and a number of observations. One example is the stability of the propagating modes imposed by forcing the square of the speed to be positive, in order to avoid imaginary frequencies, and positivity of energy, in order to avoid ghosts and therefore retain stability at a quantum level. Other possible ways to constraint the couplings is to consider the PPN expansion of the theory, and other probes of astrophysical and cosmological origin; as an example, we can mention the limits imposed by vacuum Čerenkov radiation (Elliott et al., 2005; Jacobson, 2007) and binary pulsars (see e.g. Yagi et al., 2014).

2.2.2 *Hořava gravity*

As was mentioned earlier, it has been known since a few decades that including higher than second order derivatives could help to improve the UV behaviour of a gravity theory (see Stelle, 1977). It is also known that, due to the presence of time derivatives of order higher than two, this option leads in general to a theory with ghosts. The presence of ghosts in a theory is an indication of the loss of unitarity, and therefore this solution does not seem to be an acceptable one. One could, however, attempt to modify the propagators of the theory by adding higher order spatial derivatives while retaining second order time derivatives. The shortcoming of such solution is that it will explicitely break Lorentz symmetry. On one hand, the presence of higher order spatial derivatives implies that the propagators will be modified in a way that helps to reinstate renormalisability. At the same time though higher order derivatives will modify the dispersion relations of the theory to higher order ones, thus allowing superluminal and possibly even instantaneous propagation of signals.

We can — and indeed in all the rest of this work we will — accept this possibility, and explore the consequences of such a drastic choice. In any case, since the modified behaviour of the propagator is in principle needed only in the high energy regime, it could be conceived that Lorentz invariance is recovered to an adequate degree in the low-energy regime.

Hořava-Lifshitz gravity, proposed originally in Hořava (2009a,b) (see also Sotiriou (2011) for a more recent review), is an attempt to put these arguments in a more rigorous framework. The basic idea of this theory is to retain 2 time derivatives while introducing at least $2D$ spatial derivatives — D being the spatial dimension of the theory. The reason behind this strategy is that restricting the number of time derivatives in the theory allows us to avoid the problems with the emergence of ghosts that were encountered in Stelle (1977), while at the same time increasing the number of spatial derivatives allows the modification of the propagators necessary to recover the renormalisability of the theory. In fact, one possible way to produce a sensible theory with this characteristics is to introduce some degree of anisotropic scaling between space and time dimensions in the form [see (2.2)]

$$t \to b^{-z} t \,, \qquad x^i \to b^{-1} x^i \,; \tag{2.33}$$

the resulting theory has the same structure of the Lifshitz scalar we discussed in Section 2.1 and therefore we can argue that the renormalisability properties studied before will carry to the gravitational counterpart.

What we are left to do at this point is to construct a theory with all the characteristics described above. In the first place, since we need to have more spatial derivatives than time derivatives, we need a way to separate time and space in a consistent way; to do this we will employ the Arnowitt-Deser-Misner (ADM) decomposition (Arnowitt et al., 1962, 2008). The spacetime interval will be written therefore as

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + \gamma_{ij} \left(\mathrm{d}x^i + N^i \mathrm{d}t \right) \left(\mathrm{d}x^j + N^j \mathrm{d}t \right) \,. \tag{2.34}$$

A byproduct of considering spatial and time derivatives on a different footing is that we need to choose a foliation of spacetime — which we will refer to as the preferred foliation — where t represents a *preferred* notion of time and the spatial coordinates lie on the leaves of the foliation. The action of the theory will not be symmetric under the complete group of diffeomorphisms of GR, but rather under a restricted group of transformations that preserve the foliation structure, given in their most generic form by

$$t \to \tilde{t}(t) \,, \qquad x^i \to \tilde{x}^i(t, x^i) \,. \tag{2.35}$$

We will refer to this restricted symmetry group as the *foliation preserving diffeomorphisms* ($\mathrm{Diff}_{\mathcal{F}}$).

The action of the theory is given by

$$\mathcal{S}_{\mathrm{HL}} = \frac{M_{pl}^2}{2} \int \mathrm{d}t\, \mathrm{d}^3x\, N\sqrt{\gamma} \left[K^{ij}K_{ij} - \lambda K^2 - V(\gamma_{ij}, N) \right] , \tag{2.36}$$

where K_{ij} is the extrinsic curvature of the leaves of the preferred foliation, given by

$$K_{ij} = \frac{1}{2N} \left[\dot{\gamma}_{ij} - \nabla_i N_j - \nabla_j N_i \right] ; \tag{2.37}$$

additionally λ is a running coupling, M_{pl} is a constant which we identify with the Plank mass and γ_{ij} is the spatial metric introduced in (2.34) (γ being the determinant of such spatial metric).

One thing worth noticing at this point is that, since by the definition in (2.37) K_{ij} only contains first order time derivatives, the action (2.36) can only include up to quadratic terms in the extrinsic curvature in order to limit the number of time derivatives to two.

The last thing we need to define at this point is the potential that appears in the action (2.36). As we have seen in Section 2.1, in order for the theory to be power counting renormalisable the potential will need to contain at least sixth order spatial derivatives; this also corresponds to choosing $z = 3$ in (2.33). For the time being we will adopt a conservative approach and consider sixth order as the maximum order for spatial derivatives in the potential; this way we consider the simplest possible renormalisable theory we can write. Of course, if we decide to increase the order of derivatives, the theory will remain well behaved but the number of terms we need to include will increase.

Even with restricting the order of spatial derivatives to sixth, the generic theory has quite a large number of terms that can be included in the potential (Blas et al., 2010b). One possible approach to make the theory more tractable is to find some prescription able to reduce the number of terms in the potential, thus simplifying the outcoming theory. The way we choose the potential will therefore decide which particular version of the theory we consider.

Detailed balance.

One way to choose the potential, proposed originally in Hořava (2009b), is to require that V should be derivable from a superpotential W as

$$V = E^{ij} G_{ijkl} E^{kl} \,, \tag{2.38}$$

where

$$E^{ij} = \frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta \gamma_{ij}} \quad \text{and} \quad G^{ijkl} = \frac{1}{2} \left(\gamma^{ik} \gamma^{jl} + \gamma^{il} \gamma^{jk} \right) - \lambda \gamma^{ij} \gamma^{kl} \,. \tag{2.39}$$

This prescription is called *detailed balance* and turns out to be quite useful in reducing the number of possible terms; in fact the number of terms in V is reduced to just six, with three independent couplings.

This is not a general choice however, and in Hořava (2009b) it was introduced as a purely pragmatic choice. The intuition that led to this choice though was drawn from methods commonly used in quantum critical systems, and therefore it was claimed that also in gravity there could be some conceptual reason to restrict the theory in this way.

In the few years following Hořava's proposals, many works have been published concentrating on the viability of the theory, and in particular of the detailed balance mechanism. While as we saw above, this mechanism is quite useful in making the theory tractable by reducing the number of terms in the action, some problems were uncovered.

In the first place (Vernieri and Sotiriou, 2012) one of the terms derived by the superpotential — in particular the fifth order term — is a parity violating one. In addition the scalar graviton of the theory doesn't actually produce a sixth order dispersion relation, thus threatening the renormalisability of the theory. A simple solution to both problems was nonetheless found within detailed balance, by allowing higher order (super-renormalisable) terms in the action.

A more serious problem is related to the presence of a cosmological constant. While the mere presence of one is not in general problematic, in the detail balance version of Hořava gravity the cosmological constant is not an independent parameter, but is found instead as a combination of the other parameters of the theory. As a consequence, this cosmological constant enters the theory with a negative sign and its (bare) value is unacceptably large (Nastase, 2009; Sotiriou et al., 2009a,b; Vernieri and Sotiriou, 2012). This is a more serious problem, and a solution has not been found as yet. There are claims however that this result could really be a blessing in disguise, since the presence of a vacuum energy could cancel out the cosmological constant to obtain the values observed (Appignani et al., 2010). This problem anyway is still waiting for a solution, so we will not consider it any further.

Projectable Hořava gravity.

An alternative simplification, also proposed in Hořava (2009b), consists in requiring the lapse to depend only on time, i.e. $N \equiv N(t)$. This prescription, dubbed *projectability condition*, reduces by much the number of possible operators in the action. In particular, since V can contain only spatial derivatives and the lapse no longer depends on spatial coordinates, the potential will not depend on the lapse at all but only on the spatial metric γ_{ij} and its spatial derivatives.

The resulting action for this version of the theory is given as (Sotiriou et al., 2009a; Weinfurtner et al., 2010)

$$\mathcal{S}_{\text{HL}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\text{LV}} \,, \tag{2.40}$$

where

$$\mathcal{S}_{\text{EH}} = \frac{M_{pl}^2}{2} \int \mathrm{d}t\, \mathrm{d}^3x\, N\sqrt{\gamma} \left[K^{ij}K_{ij} - K^2 + \mathsf{R} - \Lambda \right] \,, \tag{2.41}$$

and

$$\begin{aligned} \mathcal{S}_{\text{LV}} = \frac{M_{pl}^2}{2} \int \mathrm{d}t\, \mathrm{d}^3x\, N\sqrt{\gamma} \Big[& \xi K^2 - g_2 \mathsf{R}^2 - g_3 \mathsf{R}_{ij}\mathsf{R}^{ij} - g_4 \mathsf{R}^3 + \\ & - g_5 \mathsf{R}(\mathsf{R}_{ij}\mathsf{R}^{ij}) - g_6 \mathsf{R}^i{}_j \mathsf{R}^j{}_k \mathsf{R}^k{}_i - g_7 \mathsf{R}\nabla^2\mathsf{R} - g_8 \nabla_k \mathsf{R}_{ij} \nabla^k \mathsf{R}^{ij} \Big] \,. \end{aligned} \tag{2.42}$$

A few comments here are in order: in the first place, in the original proposal (Hořava, 2009b) detailed balance was imposed on top of projectability. Imposing projectability alone though doesn't make the theory too complicated to use, as we saw in (2.40), and for this reason (and given the problems that detailed balance introduces) it could be a possibility to just impose projectability. As a side note, detailed balance without projectability was considered in Vernieri and Sotiriou (2012) but, while some of the problems that plague the version of the theory originally proposed are solved, the cosmological constant problem is untouched; this possibility doesn't seem therefore a worthy path to take.

The projectability condition does in fact help in fixing some problems encountered with detailed balance. Parity invariance can be imposed in this version of the theory, and therefore we don't need to worry anymore with the presence of parity violating terms. In addition, the cosmological constant in this case is a free parameter and can be fixed by observations.

One major problem was uncovered though in this version of the theory. It was noticed in fact (Blas et al., 2009; Charmousis et al., 2009; Koyama and Arroja, 2010; Weinfurtner et al., 2010) that the GR limit $\lambda \to 1$ in (2.36) (equivalently the $\xi \to 0$ limit in (2.40)) presents some problems, namely a gradient instability at low energies accompanied by strong coupling. This behavior emanates from the kinetic part of the action: the solution of the momentum constraint yields a shift vector with longitudinal component $B \propto (\lambda - 1)^{-1}$. As the perturbative expansion of the action contains arbitrary powers of B, upon canonical normalization, terms of higher order acquire coefficients with increasing powers of the factor $(\lambda - 1)^{-1}$. This way, if in the IR $(\lambda - 1)$ runs to sufficiently small values from above, the perturbative expansion that led to the conclusion that there is an instability actually breaks down.

This analysis on the other hand leaves open the possibility of a non-perturbative restoration of the GR limit. Indeed, there are indications that $\lambda \to 1$ limit could continuously connected to GR for spherically symmetric configurations (Mukohyama, 2010) and for cosmological solutions (Izumi and Mukohyama, 2011; Gümrükçüoğlu et al., 2012). This approach might present us with yet more problems on the other hand, first and foremost the fact that the renormalisability was argued on a perturbative basis and therefore a non-perturbative solution might threaten the very reason the theory was introduced in the first place.

Non-projectable Hořava gravity.

As we discussed above, all the restrictions to Hořava theory that have been proposed until now are plagued with problems. One possible alternative is therefore that of abandoning the idea of restricting the theory, and accepting the most generic action we can write.

The most generic version of Hořava theory is therefore the one in which we don't enforce any condition on the potential. This case is called *non-projectable* and will contain all the possible terms compatible with the symmetries of the theory (namely with the restricted diffeomorphism invariance $\mathrm{Diff}_{\mathcal{F}}$ introduced in (2.35)). Also, as was first pointed out in Blas et al. (2010b), as the lapse depends on both space and time we need to include in the potential also the terms constructed with $a_i = \partial_i \ln N$ — both the contractions of the type $a_i a^i$ (including powers of this term) and the contractions of a_i with the tensors built out of the metric. The lowest order invariant we can build, which comes at the same order as the spatial Ricci scalar R, is $a^2 = a_i a^i$.

The action in this case will then be given by

$$\mathcal{S}_{\rm HL} = \frac{M_{pl}^2}{2} \int \mathrm{d}t\, \mathrm{d}^3x\, N\sqrt{\gamma} \left[K_{ij}K^{ij} - \lambda K^2 + \xi \mathsf{R} + \eta a^2 + \frac{1}{M_A^2}\mathcal{L}_4 + \frac{1}{M_B^4}\mathcal{L}_6 \right] , \tag{2.43}$$

where this time $\mathcal{L}_4$ and $\mathcal{L}_6$ include all possible operators of order 4 and 6 in spatial derivatives one can build using γ_{ij} along with its spatial derivatives and a_i. The number of operators present in $\mathcal{L}_4$ and $\mathcal{L}_6$ is quite large, of order 10^2.

As a final remark, the analysis of perturbations around flat spacetime reveals that the propagating degrees of freedom are one transverse traceless tensor (spin-2) graviton and one scalar degree of freedom (spin-0), also often referred to as a scalar graviton.

2.2.3 *Covariant Hořava gravity and the relation to Æ theory*

One of the most interesting aspects of Hořava gravity is the fact that its low-energy limit, i.e. the one given by the action of (2.43) without the higher order terms in $\mathcal{L}_4$ and $\mathcal{L}_6$, turns out to be equivalent to a restricted version of Einstein-Æther theory (Jacobson, 2010) when written in a covariant way. It will be interesting to see in some detail how this work, as it will introduce some notations and conventions which will turn out to be useful in the following.

Let's consider first Einstein-Æther theory. Since we want to compare this theory to Hořava gravity, we will have to enforce a foliation in the theory; this can be done easily by restricting the æther vector to be hypersurface orthogonal. We will do this by writing the æther as a function of a scalar field T which we will assume to define the foliation as the level surfaces of the scalar field. The æther vector will therefore be expressed as

$$u_a = -N\nabla_a T \; , \tag{2.44}$$

where the lapse is given as

$$N^{-2} = -\mathrm{g}^{ab}\nabla_a T \nabla_b T \; . \tag{2.45}$$

From this last expression, it's easy to see that the expression for N is chosen in the appropriate way so that the normalization $u^2 = -1$ holds automatically. Also, since we are trying to express the theory in a foliated spacetime, it can be useful to decompose the metric as

$$\mathrm{g}_{ab} = -u_a u_b + \mathrm{p}_{ab} \; ; \tag{2.46}$$

in this decomposition, p_{ab} represents the 3D metric on the leaves of the foliation, and can be used as a projector on the foliation. Additionally, due to the hypersurface orthogonality of the æther together with the unit norm constraint, which implies that the twist of the æther vanishes, we may decompose the derivatives of the æther as

$$\nabla_a u_b = -u_a a_b + K_{ab} , \tag{2.47}$$

where as defined before $a_a = u^b \nabla_b u_a$ is the acceleration of the æther and K_{ab} is the extrinsic curvature of the leaves of the foliation, defined as

$$K_{ab} = \frac{1}{2} \pounds_u \mathsf{p}_{ab} . \tag{2.48}$$

Here and in the following, recalling a rather standard notiation, $\pounds_u$ will identify the Lie derivative along the æther vector u^a.

Using the results listed until here, including the fact that in this formulation $u^2 = -1$ is implied automatically by the definition of u_a and that both K_{ab} and a_a are purely spatial (i.e. $u^a K_{ab} = u^a a_a = 0$), we find that the æther lagrangian (2.27) is reduced to

$$\mathcal{L}_u = -c_{13} K_{ab} K^{ab} - c_2 K^2 + c_{14} a^2 , \tag{2.49}$$

where we introduced the convention (which will also be used throughout the thesis) $c_{lm} = c_l + c_m$. If we finally use the Gauss-Codazzi equations

$$\mathcal{R} = K_{ab} K^{ab} - K^2 + \mathsf{R} - 2\nabla_a \left(K u^a - a^a \right) \tag{2.50}$$

to decompose the Ricci scalar along the foliation, we can easily see that the æther lagrangian (2.27) is exactly the same as the low-energy version of the Hořava lagrangian used inside (2.43) when we identify the couplings as

$$\xi = \frac{1}{1 - c_{13}} , \quad \lambda = \frac{1 + c_2}{1 - c_{13}} , \quad \eta = \frac{c_{14}}{1 - c_{13}} . \tag{2.51}$$

The importance of this result is twofold. In the first place, despite being a different theory, we see that the reduced version of Einstein-Æther theory where the æther is assumed to be hypersurface orthogonal at the level of the action is at all effects equivalent to the low-energy limit of Hořava theory. This means in particular that solutions of hypersurface orthogonal Einstein-Æther theory are also solutions of low-energy Hořava gravity (Jacobson, 2010); the converse is not generically true. This fact turned out to be extremely useful because a number of solutions of Einstein-Æther theory — in particular the spherically symmetric ones, where the æther is automatically hypersurface orthogonal — are also solutions of Hořava gravity, thus removing the necessity of computing them anew.

The second important aspect of this equivalence is that the lagrangian given in (2.26), considered together with the definitions of (2.44) and (2.45), can be considered a lagrangian for low-energy Hořava gravity. This way, we have a version of the Hořava lagrangian formulated in a covariant way, without the need to resort to the ADM variables. This will turn out to be quite handy in the following.

As a last comment, it's worth mentioning that this equivalence also provides an easy explanation for the difference in the number of propagating modes of the two theories. Indeed as we saw before, in Einstein-Æther theory the propagating modes are five, namely two tensor modes, two vector modes and one scalar mode, while in Hořava gravity there are just three, namely the two tensor modes and the scalar mode. It is clear that assuming hypersurface orthogonality will kill off the two transverse vector modes, and therefore this restricted version of Einstein-Æther theory has a reduced number of propagating modes, which match with the propagating modes of Hořava gravity as we expect from the equivalence of this last two theories.

♦

As we have discussed in this Chapter, these are the premises for a viable theory of gravity that makes sense in a quantum world. This theory on the other hand is not free of problems, and before celebrating there is a lot of work left to do. The goal of this thesis is exactly this: to contribute —

however small the contribution — to expanding the knowledge on Lorentz violating theories.

When it comes to Lorentz violating theories of gravity, three main issues come to mind. The first issue is that of renormalisability. We have seen that Lorentz breaking can in principle help with the renormalisability of QFTs, and we have as well seen that the main candidate within this type of theories for a quantum gravity theory, which is Hořava theory, does at a first sight satisfy the requisites for being at least power-counting renormalisable. It is true on the other hand that in general power-counting is just an indication of a theory being actually renormalisable and hence it cannot be considered as a final proof of the viability of the theory. Various studies have been therefore undertaken in the last few years, to better understand the renormalisation properties of Hořava gravity; all this works are still performed in the restricted setting of projectable Hořava theory in order to be able to perform the extremely complicated calculations, but there is hope that the results will extend to the full theory. Examples of such works can be found in Iengo, Russo, and Serone (2009); Contillo, Rechenberger, and Saueressig (2013); Benedetti and Guarnieri (2014). Interestingly enough, quite recently a proof of the renormalisability at all orders of projectable Hořava gravity has been produced (Barvinsky et al., 2016).

The second problem that all Lorentz violating theories of gravity face is that of coupling the gravitational theory to the matter sector in a meaningful way. As we mentioned before (see for details Kosteleky and Russell, 2011), Lorentz violations in the matter sector are extremely tightly constrained by particle physics experiments. For this reason, whenever we consider a gravity sector that violates Lorentz symmetry, we need to find a mechanism to prevent Lorentz violations to percolate to the matter sector through quantum corrections. This problem is still open, with only a few proposals for its resolution; a detailed discussion of this can be found in Pospelov and Shang (2012).

The last issue that we need to discuss is that related to the existence of black holes. As is known, black hole solutions in general relativity rely heavily on the causal structure of the spacetime for their existence. The properties of the causal structure in turn rely on Lorentz symmetry, and in particular on the existence of a maximal speed of propagation given by the speed of light c. If Lorentz symmetry is violated then superluminal, and even instantaneous, propagation of signals in a particular frame becomes a distinct possibility. This leads us to question the very existence of black holes, and in case of an affirmative answer, to the dynamics of their formation.

Surprisingly enough, black hole solutions in Lorentz violating gravity theories have been found (see e.g. Eling and Jacobson, 2006; Barausse et al., 2011; Sotiriou et al., 2014), and the dynamic of their formation has been studied (Garfinkle et al., 2007), at least in some restricted setting. On the other hand, many properties of such objects, and even rigorous definitions as such, are still missing.

In this thesis we will concentrate our efforts on the last two aspects we just discussed, in the attempt to discuss the problems in more depth and to provide some solution to them. The structure of the thesis will be as follows. In Chapter 3 we will discuss in detail the problems encountered when adding matter to the theory and we will try to discuss the properties of a particular solution to the problem. In Chapter 4 and 5 we will discuss how to define in a rigorous way the causal structure of spacetimes with a preferred foliation, and how to define black holes in such settings. We will also discuss a number of interesting properties of these objects. Finally in Chapter 6 we will discuss some aspects of the dynamics of gravitational collapse leading to the formation of black holes in Lorentz violating theories, and in Chapter 7 we will draw the conclusions of the work done, and we will discuss some leads for future work.

3

ADDING MATTER TO THE THEORY

In the long quest for a quantum gravity theory, Lorentz violations might play an important role. As we have seen in Chapter 2, they can help to improve the renormalisability properties of quantum field theories; in particular a quantum theory of gravity that is potentially renormalisable, such as Hořava gravity, could be finally within reach. In fact, Hořava gravity is gaining ground as a possible candidate and more and more people are concentrating their efforts in studying its properties.

On the other hand there is an element of caution that need to be exercised whenever we choose to consider Lorentz violations in a gravity theory. Lorentz symmetry is constrained quite weakly in the gravity sector,[1] and this is exactly what allows us to write a Lorentz breaking gravity theory in the first place. On the other hand, in the matter sector Lorentz symmetry is constrained to a quite high degree of precision (Mattingly, 2005; Kosteleky and Russell, 2011; Liberati, 2013); for this reason whenever we choose to consider a Lorentz violating gravity theory, we have to ask ourselves how to avoid the percolation of Lorentz violations from the gravity to the matter sector. In general the Lorentz breaking operators could percolate to the matter sector through quantum corrections, and therefore we need to endow the theory with some mechanism to protect the matter sector from this eventuality.

1 Stricter constraints can be obtained using the data derived from gravitational wave detections. For more details see e.g. Hansen et al. (2015)

This issue plagues Lorentz violating theories in general, and Hořava gravity is not immune to it. A good deal of effort has been put into trying to find ways to avoid this kind of ill behaviour, and many protecting mechanisms have been proposed. A first possibility is that of employing some custodial symmetry that will prevent Lorentz violating operators from appearing in the matter sector. The best known symmetry able to do so is supersymmetry (SUSY). There are proposals to employ such symmetry to solve this problem (Nibbelink and Pospelov, 2005; Xue, 2010; Redigolo, 2012; Pujolas and Sibiryakov, 2012), but it's not clear if a supersymmetric version of Hořava gravity can exist in the first place. A similar solution was proposed in Pospelov and Tamarit (2014), consisting in coupling a supersymmetric matter sector to a Lorentz violating non-supersymmetric gravity sector; doing so would naturally suppress the Lorentz violating operators in the matter sector. There is not much work done on this proposal though, and it's therefore unclear whether and how this could be implemented in practice.

An alternative possibility is to use strongly coupled systems; it was shown in this case (Chadha and Nielsen, 1983; Bednik et al., 2013; Kharuk and Sibiryakov, 2015) that the running of the couplings of Lorentz violating operators present in the high energy regime will force the couplings themselves to vanish in the low-energy regime, therefore re-establishing Lorentz symmetry as an emergent symmetry rather than a fundamental one. As before, more work is required in order to assess if this last solution is a viable one.

The possibilities discussed just above are quite interesting, and deserve in depth investigation. There is however a third possibility, which will be discussed in some detail in the present Chapter. Since we are chiefly interested in this last option, and we won't consider any longer the ones mentioned above.

Said third option is offered by a separation of scales. A proposal for a solution to the problem in this direction was studied in Pospelov and Shang (2012); in this work it was shown how separating the scale below which

Lorentz symmetry in gravity is restored and the high energy (Plank) scale can bring the percolation of Lorentz violating operators below experimental constraint. At the same time, a problem was uncovered: some quadratic divergences were found in a sector of the theory, thus creating a naturalness issue.

In this Chapter we will first examine how this mechanism works, and explain in more detail the issues that were found. We will then go on to examine a possible solution to the naturalness problem mentioned above, and we will try to understand the impact said solution might have on the overall theory, both from the renormalisability and the stability point of view.

3.1 POWER SUPPRESSED INTERACTIONS AND THE MIXED DERIVATIVES CONUNDRUM

In order to find a way to protect a theory from Lorentz violations percolating from the gravity to the matter sector (which we assume here to be the Standard Model), let's consider a generic Lorentz violating gravity sector that couples to the SM sector via a power suppressed interaction given by

$$\frac{1}{M^{n+k-4}}\mathcal{O}_{LV}^{(n)}\mathcal{O}_{SM}^{(k)}\,, \tag{3.1}$$

where $\mathcal{O}_{LV}^{(n)}$ and $\mathcal{O}_{SM}^{(k)}$ are operators of dimension n and k respectively, M represents a (very) high energy scale and the condition $n+k \geqslant 5$ holds.

Being power suppressed, this operator would typically produce some power-divergent loop integral (Pospelov and Shang, 2012). As an example, for the case $n = 1$ and $k = 4$ we can — using a standard technique of Effective Field Theories — eliminate the fields in the high energy Lorentz violating sectors by "integrating out" of the action the corresponding high energy operators; the impact this will have in the low-energy sector though is that of producing Lorentz violating terms in the SM sector, given as

$$\frac{1}{M}\mathcal{O}_{LV}^{(1)}\mathcal{O}_{SM}^{(4)} \quad \rightarrow \quad \frac{\Lambda_{UV}^2}{M^2}\mathcal{O}_{SM,LV}^{(4)}\,, \tag{3.2}$$

with Λ_{UV} being the scale above which the high energy operators appear.

Theories of this kind are normally not considered on phenomenological grounds, since this operator will be of order 1; this happens because the high energy scale is normally considered to be at the same order as the Plank scale, i.e. $\Lambda_{UV} \sim M$. Such high degree of Lorentz violation is not acceptable since, as mentioned before, it is hard to accomodate within the constraints on Lorentz violations in the SM sector (Kosteleky and Russell, 2011).

An interesting possibility would arise if the loops in the Lorentz violating sector were stabilised by some mechanism at high energy, in such a way that Λ_{UV} is replaced by some physically meaningful scale that is well below M. In this case the Lorentz violating contribution of (3.2) can be made arbitrarily small (Pospelov and Shang, 2012). A known example is that of introducing higher order derivatives in the theory, as a way to improve the convergence of loop integrals. Hořava gravity exhibits this behaviour.

In Pospelov and Shang (2012) the idea above was applied to Hořava gravity, and it was shown by calculating the one-loop contributions to the vector and scalar modes in Hořava-type theories, that the size of the induced Lorentz violating terms in the matter sector is controlled by the ratio Λ_{HL}^2/M^2. Lorentz violations in the matter sector could therefore be considered under control, as long as the scale Λ_{HL} below which Lorentz symmetry is restored in the gravity sector is sufficiently smaller than the Plank scale M.

One issue that was uncovered in Pospelov and Shang (2012) though was that in the vector sector, some residual quadratic divergences remained. This behaviour creates a naturalness problem which threatens the consistency of the theory; we need therefore to find a way to avoid this kind of behaviour in the vector sector.

A possible solution to such naturalness problem was already proposed in Pospelov and Shang (2012). The addition to the Hořava action of a single term is sufficient to suppress all the quadratically divergent contributions to the loop integral of the vector modes, and therefore will render the loop-induced Lorentz violations in the matter sector completely under control.

This very simple solution presents on the other hand a conundrum. The term that was added to the action has the form $\alpha \nabla^i K_{ij} \nabla_k K^{kj}$; employing the scaling dimensions usually adopted in Hořava gravity this term appears to have dimension 8 and should therefore be ruled out. On the other hand this term, thanks to the presence of the time derivatives, becomes the leading order kinetic term in the UV; for this reason, the power counting scheme and the scaling dimension of the theory will have to necessarily change (Colombo et al., 2015a).

Two possible issues arise from such a drastic modification of the theory. In the first place, while this type of term can indeed cure the quadratic divergences in the vector sector of the theory, changing the scaling and hence the power counting scheme could irremediably spoil the renormalisability and the stability properties of the theory as a whole. Additionally, even in the eventuality that the theory remained well behaved with the new scaling, there would be new terms that enter the action at the same order as the term mentioned above, and which we would have no reason to disregard. The impact such new terms could have though is something we will have to understand. The remainder of this Chapter will then be devoted to investigating these two issues.

3.2 IS IT RENORMALISABLE? LIFSHITZ SCALAR CAN HELP

The first and foremost issue we encounter whenever we consider adding terms to Hořava gravity is that of renormalisability. The theory in its original formulation is definitely promising as far as renormalisability is concerned (see Section 2.1 of Chapter 2 for further details), but we would like to make sure that the addition of the extra terms mentioned above won't spoil such good behaviour. The answer to this question is what we are after in this Section.

Once again we will employ the Lifshitz scalar theory to address this question, this time adding a term mimicking the mixed derivative term proposed in Pospelov and Shang (2012); we will hence re-analyse the renormalisability properties of the theory under this new perspective. A preliminary treatment of this aspect was given in Colombo et al. (2015a), while a much more thorough and dedicated analysis of this particular case was reported in Colombo et al. (2015b).

In order to consider the mixed derivative case, we expand the lagrangian previously used for the Lifshitz scalar (2.1) by including a mixed term; the new lagrangian will then look like

$$\mathcal{L} = \alpha\, \dot{\phi}^2 + \beta\, \dot{\phi}(-\triangle)^y \dot{\phi} - \gamma\, \phi(-\triangle)^z \phi \;. \tag{3.3}$$

The anisotropic scaling is the same we used before, given as

$$t \to b^{-m} t \;, \qquad x^i \to b^{-1} x^i \;. \tag{3.4}$$

This time though we won't assume $z = m$ as we did in the case of the plain Lifshitz scalar, but we will rather derive some condition on z later on.

The dimensions of the coupling constants in (3.3) are related through

$$[\alpha] = [\beta][\kappa]^{2y} \;, \qquad [\gamma] = [\beta][\kappa]^{2(m+y-z)} \;. \tag{3.5}$$

We can then rewrite the lagrangian (3.3) as

$$\mathcal{L} = \beta\left[\xi\, M^{2\,y} \dot{\phi}^2 + \dot{\phi}(-\triangle)^y \dot{\phi} - M^{2\,(m+y-z)}\, \phi(-\triangle)^z \phi\right] \;, \tag{3.6}$$

where $[M] = [\kappa]$ and $[\xi] = [\kappa]^0$.

Here, we choose the normalization such that $\beta = 1$ and fix the units such that the coupling constants for the last two terms, which are expected to dominate in the UV, have the same dimensions. The latter condition gives the relation

$$m = z - y \;, \tag{3.7}$$

which, assuming a theory with fixed y and z, determines the degree of anisotropic scaling.

Requiring the action

$$\mathcal{S} = \int dt\, d^D x\, \mathcal{L}\,, \tag{3.8}$$

to be dimensionless, the dimension of the Lifshitz scalar can be computed to be

$$[\phi] = [\kappa]^{d_\phi} = [\kappa]^{(D-m-2y)/2}\,. \tag{3.9}$$

If the scalar field ϕ is dimensionless or has negative dimension, i.e. $d_\phi \leqslant 0$, the coupling constants of ϕ^n interactions with arbitrary positive integer n has positive dimensions. As we already saw for the simplest case, the standard lore in QFT dictates that positive dimensional coupling constants is an indication of renormalisability for the corresponding interactions. This translates into the condition

$$z = m + y \geqslant D - y\,. \tag{3.10}$$

This condition was found in Colombo et al. (2015a) for the particular case $y = 1$. Here (see Colombo et al., 2015b) we expanded the treatment to the case with generic y.

At this point we have to note that the dimensional arguments we presented until now have to be treated with caution. Indeed, as we will show in the next Section, they cease being trustworthy once derivative interactions are taken into account. On the other hand, we are ultimately interested in using the Lifshitz scalar as a proxy for understanding the UV properties of a gravity theory with the same anisotropic scaling properties and derivative structure. In a gravity theory derivative interactions are inevitable and therefore as a next step we will include this type of interactions in the Lifshitz scalar theory and we will try to derive a somewhat more robust criterion to show the renormalisability of the theory. This criterion is, as we used previously, the superficial degree of divergence.

3.2.1 *Superficial degree of divergence for derivative interactions*

We now consider the UV limit (i.e. $k \gg \xi^{1/2y} M$) of the free theory by choosing the appropriate normalization and units [see (3.7)] in (3.6) to obtain

$$\mathcal{L}_{\text{UV}} = \dot{\phi}(-\triangle)^y \dot{\phi} - \phi(-\triangle)^z \phi \,. \tag{3.11}$$

The Green's function for the Lifshitz scalar can be computed as

$$G(\omega, \boldsymbol{k}) \propto \frac{1}{k^{2y}\left[\omega^2 - k^{2\,m}\right]} \,. \tag{3.12}$$

Since, as before, we are interested in finding the dimension of a generic Feynman diagram, internal lines in the diagram contribute as

$$G(\omega, \boldsymbol{k}) \to [\kappa]^{-2(m+y)} = [\kappa]^{-2z} \,. \tag{3.13}$$

Each loop integral contributes through the momentum space volume element as

$$\int \mathrm{d}\omega \, \mathrm{d}^D k \;\to [\kappa]^{m+D} = [\kappa]^{z+D-y} \,. \tag{3.14}$$

We will consider the most general self-interaction term given by

$$\mathcal{L}_{\text{int}} = \lambda \left(\nabla_i^{p_x}, \partial_t^{p_t}, \phi^s\right) , \tag{3.15}$$

where λ is the coupling constant, while $(\nabla_i^{p_x}, \partial_t^{p_t}, \phi^s)$ is shorthand for an s–particle operator that contains p_x spatial derivatives, p_t temporal derivatives, or $p \equiv p_x + m\, p_t$ weighted derivatives. The dimension of the coupling constant can then be found as

$$[\lambda] = [\kappa]^{d_\lambda} = [\kappa]^{D+m-p-s\,d_\phi} \,. \tag{3.16}$$

Assuming that all the derivatives in a given vertex V arise from internal lines, the contribution from each vertex will have dimension

$$[V] = [\kappa]^p = [\kappa]^{p_x + m\, p_t} \,. \tag{3.17}$$

In conventional field theory, it is typically sufficient to have a finite number of interactions that are renormalisable. However, here we are actually using a scalar field theory as a toy theory that will give us some insight into the renormalisability properties of a Lorentz-violating gravity theory. The perturbative expansion of a gravity theory includes infinitely many terms, due to the perturbative expansion of the inverse metric. All of these terms should be renormalisable for the theory to have the desired UV behaviour. Therefore what we need to require is that any interaction of the type (3.15), with $s \to \infty$, be renormalisable. We purposefully avoid choosing any particular term from some specific theory as an example, as the renormalisability of any such term would not necessarily imply that the gravitational analogue is renormalisable.

For a diagram with L loops, I internal lines, E external lines and V vertices, the superficial degree of divergence is calculated as[2]

$$\delta \leqslant L(D+m) - 2I(m+y) + V\,p\,. \tag{3.18}$$

Using two well-known identities, stemming from general properties of Feynman diagrams

$$L - I + V = 1\,, \qquad s\,V = E + 2\,I\,, \tag{3.19}$$

we can extract more information from the superficial degree of divergence. To do so we first eliminate L and I using (3.19) to find

$$\delta \leqslant D + m - d_\phi E - d_\lambda V\,, \tag{3.20}$$

where as above d_ϕ and d_λ are the dimensions of the field and of the coupling constant, respectively.

This result is compatible with the standard intuition for power counting renormalisability: provided that $d_\phi > 0$, any interaction with positive dimension coupling constant $d_\lambda > 0$ will lead to a small and finite number or zero divergent diagrams, as convergence improves when the number of

2 The assumption that all the momentum contributions at a given vertex comes from internal lines is a conservative one. Instead, if one imposes shift symmetry $\phi \to \phi + c$, all the external lines (E) would be associated with at least one spatial derivative of the field, contributing $-E$ to the right-hand side of (3.18).

vertices or the number of external lines is increased. The condition $d_\lambda > 0$ can then be interpreted as an upper bound on s and p, by employing the expression in (3.16).

When one wishes to use the Lifshitz scalar as a proxy for the behaviour of a gravity theory with the similar derivative structure, this standard result is not particularly useful. Gravity theories are highly nonlinear and expanding around a given background will generate an infinite number of terms with infinite copies of the field, albeit the limited number of derivatives in each term. Hence, one would wish to have convergent diagrams for any value of s. It is clear that this can only be achieved if $d_\phi \leqslant 0$.

Equation (3.20) is not very helpful when considering the $d_\phi < 0$ case, as the external lines contribution comes with the wrong sign. However, using the identities in (3.19) one can rewrite (3.18) as

$$\delta \leq 2\,z + 2\,d_\phi\,L - (2\,z - p)\,V\ . \tag{3.21}$$

It is now straightforward to see that, so long as $d_\phi \leqslant 0$, the contribution from the loop either vanishes or each loop contributes with more negative powers of the cutoff. It is the number of vertices, or more specifically the number of weighted derivatives in a vertex that really determine how divergent the diagram is. For example, for non-derivative interactions $p = 0$, we see immediately that the degree of divergence is $\delta \leqslant 0$ if $d_\phi < 0$, indicating that ϕ^n are either log divergent or finite (Visser, 2009a). For $0 < p \leqslant 2z$ the vertices contributions to the degree of divergence are negative, making δ bounded from above by a finite value. In other words, for the interaction terms that have equal or less weighted derivatives than the free terms, there is a finite amount of counterterms that remove the divergences. Interaction terms with $p > 2z$ will be non-renormalisable, as at a given loop order one can always have diagrams with an arbitrary number of vertices. Hence, such terms are not expected to be generated by radiative corrections.

To summarize, when derivative interactions are considered, in addition to (3.10), we obtain the second renormalisability condition which restricts the allowed number of derivatives in a given interaction

$$2\,z \geqslant p = p_x + m\,p_t \,. \tag{3.22}$$

The maximum number of spatial gradients a renormalisable interaction can have is

$$p_{x,\mathrm{max}} = 2z \,, \tag{3.23}$$

while the maximum number of time derivatives we can allow is

$$p_{t,\mathrm{max}} = \frac{2z}{m} = 2 + \frac{2y}{m} \,. \tag{3.24}$$

We have thus found that the criterion for the renormalisability of an interaction term is related to the number of derivatives it contains, rather than to its dimensions. For the case where $d_\phi = 0$, the two criteria coincide as one can already see using (3.20); a term with a positive coupling constant necessarily contains less or equal derivatives than the free theory, thus is expected to be renormalisable. However, in the case of $d_\phi < 0$, the intuitive description that links renormalisability with the dimensions of the coupling constant breaks down. For instance, if d_ϕ is "negative enough", $\nabla_i\phi$ can have negative dimensions and one can construct interaction terms with an arbitrary number of derivatives while still having a positive dimension coupling constant. Nonetheless, as we have shown above, the interaction terms with $p > 2z$ would not be renormalisable.

3.2.2 *Restrictions from predictivity and unitarity*

The last point made in the previous Section, regarding the fact that interaction terms with $p > 2z$ are non-renormalisable even though they have a positive dimension coupling constant, touches upon the issue of predictivity. If $d_\lambda > 0$ were a sufficient condition for renormalisabilty for derivative interactions, then radiative corrections would generate an infinite number of counterterms. A similar issue exists for interactions with $p < 2z$ and a large number of copies of ϕ: so long as ϕ has zero or negative dimensions, and for a given number of derivatives, there is an infinite number of renormalisable interaction terms with ever increasing copies of ϕ that do not carry derivatives. This has already been pointed out in Fujimori et al. (2015) for the $y = 0$ and $z = D$ theory, but our analysis reveals that it actually is a quite generic feature for theories with $d_\phi \leqslant 0$. One need not worry about this problem for the Lifshitz scalar with no derivative interactions because it is a finite theory. But once derivative interactions are included the existence of infinite potential counterterms poses an actual threat for predictivity. A simple solution is to impose some symmetry, e.g. a shift symmetry $\phi \to \phi + c$, thus rendering the number of terms finite (Fujimori et al., 2015).

In a gravity theory one expects to have such a symmetry anyway. In Hořava gravity in particular, the symmetry given by the foliation preserving diffeomorphisms (2.35) comes to the rescue. Although the expansion of the $\mathrm{Diff}_{\mathcal{F}}$ invariant terms lead to an infinite number of terms with ever increasing powers of the metric perturbations, the coefficients of these terms are not actually independent and can be expressed in terms of the original coupling constant; this way the number of coupling constants remain finite.

We shall now turn our attention to unitarity. In a theory with derivative interactions one has to make sure that threatening terms such as $\ddot{\phi}^2$ will not be generated by radiative corrections. The second renormalisability condition (3.22) implies that the total number of time derivatives a term can have is given by (3.24), which can be larger than 2 if $y > 0$. In fact, the simplest example with $y = 1$ and $m = 1$ studied in Colombo et al. (2015a) allows for dangerous terms with 4 time derivatives and is thus non-unitary. As we can see from (3.7), the value of m can be increased by including gradient terms with higher z in the free theory. According to (3.24), in order to avoid the unitarity breaking terms for $y > 0$, it is sufficient to require

$$m > y \,. \qquad (3.25)$$

Let us then collect all of the conditions we have derived so far. For the theory given by the lagrangian density

$$\mathcal{L} = \dot{\phi}(-\triangle)^y\dot{\phi} - \phi(-\triangle)^z\phi + \lambda\left(\nabla_i^{p_x}, \partial_t^{p_t}, \phi^s\right), \tag{3.26}$$

the power counting renormalisability and unitarity requirements lead to the set of conditions

$$\begin{aligned} z &= m + y\,, \\ m &\geqslant D - 2y\,, \\ 2z &\geqslant p_x + m p_t\,, \\ m &> y\,. \end{aligned} \tag{3.27}$$

For the case with $y = 0$, we see from (3.27) that $z = m$ and the standard renormalisability condition of Hořava gravity is recovered

$$z \geqslant D\,, \tag{3.28}$$

along with the trivially satisfied condition $m > 0$.

For the case with $D = 3$ and $y = 1$, we obtain

$$z = m + 1\,, \qquad m > 1\,, \tag{3.29}$$

where the last condition forbids relativistic scaling on the grounds of unitarity.

In Figure 1, we show the allowed (m, y) region for $D = 3$. For a given mixed derivative term with arbitrary spatial derivatives, one can always satisfy the power counting renormalisability and the unitarity conditions, provided that enough powers of gradient terms are included in the free action.

3.3 MIXED DERIVATIVE EXTENSION OF HOŘAVA GRAVITY

At this stage, we have good reason to believe that adding mixed derivatives, along with the modifications to the power counting that comes along with it,

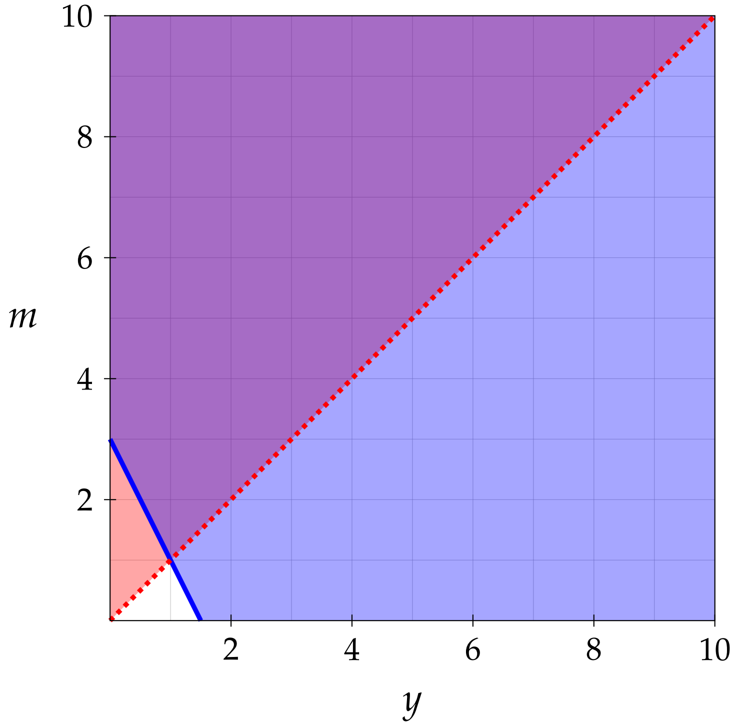


Figure 1: For $D = 3$, the allowed region for the scaling exponent and mixed derivatives. The region above (including) the solid blue line corresponds to the region where the renormalisability condition (3.10) holds. The region above (excluding) the dotted red line corresponds to the region where higher order time derivative terms are not generated (3.25). The combined allowed region is the darkest (purple) region.

will not spoil irreparably the renormalisability and unitarity of the theory; it is then time to try to understand what impact this kind of terms will have in the actual gravity theory. In particular, we would like to make sure that the propagating degrees of freedom of the theory won't introduce any additional problems.

To do this, in the rest of this Chapter, we will focus on an extension of the Diff$_{\mathcal{F}}$ invariant (2.35) non-projectable version of Hořava gravity in 3+1 dimensions. The first thing we need to do is to determine the most general action suitable for our purposes. Formally, the action we consider is given by

$$\mathcal{S} = \frac{M_{pl}^2}{2} \int \mathrm{d}t\, \mathrm{d}^3x\, N\sqrt{\gamma}\, \left(K_{ij}K^{ij} - \lambda K^2\right) + \mathcal{S}_{\mathrm{v}} + \mathcal{S}_{\times}\ , \tag{3.30}$$

where the extrinsic curvature K_{ij} is defined as before (2.37) as

$$K_{ij} \equiv \frac{1}{2N}\left(\dot{\gamma}_{ij} - \nabla_i N_j - \nabla_j N_i\right)\ . \tag{3.31}$$

The action (3.30) contains explicitely the time-derivative kinetic terms for the 3-metric γ_{ij}; the potential part of the action is given by

$$\mathcal{S}_{\mathrm{v}} = \frac{M_{pl}^2}{2} \int \mathrm{d}t\, \mathrm{d}^3x\, N\sqrt{\gamma}\, \left(\mathcal{L}_{z=1} + \frac{1}{M_*^2}\mathcal{L}_{z=2} + \frac{1}{M_*^4}\mathcal{L}_{z=3}\right)\ , \tag{3.32}$$

and contains up to 6 spatial derivatives and exhausts all marginal and relevant operators. In (3.32) M_* represents the scale above which Lorentz symmetry is broken.

$\mathcal{S}_{\times}$ denotes all terms that are compatible with the symmetry and contain up to two time derivatives and two spatial derivatives, including the mixed-derivative term considered in Pospelov and Shang (2012). One could also add the relevant deformation $\mathcal{L}_{z=0} = \Lambda$, representing the cosmological constant, which is allowed by the $\mathrm{Diff}_{\mathcal{F}}$ symmetry and the power counting. However, since we will later focus on a Minkowski background for the stability analysis, we will neglect this term.

The number of all possible terms in $\mathcal{S}_V$ and $\mathcal{S}_{\times}$ is of order 10^2. Since we are interested in linear perturbations around flat spacetime though, we can consider without loss of generality only the terms that give non-trivial contributions to the propagation of linear perturbations around the Minkowski background. We expand the basic quantities as

$$N = 1 + \delta N\ , \qquad N_i = \delta N_i\ , \qquad \gamma_{ij} = \delta_{ij} + \delta\gamma_{ij}\ , \tag{3.33}$$

and impose a truncation of the action at quadratic order in perturbations. The building blocks for constructing the $\mathrm{Diff}_{\mathcal{F}}$ invariant potential terms are the acceleration 3-vector, which contains 1 spatial derivative,

$$a_i \equiv \partial_i \log N = \partial_i \delta N + \mathcal{O}(\mathrm{pert}^2)\ , \tag{3.34}$$

and the 3 dimensional Ricci curvature tensor, containing 2 spatial derivatives

$$\mathsf{R}_{ij} = -\frac{\delta^{lm}}{2}\left[\partial_l\partial_m\delta\gamma_{ij} + \partial_i\partial_j\delta\gamma_{lm} - 2\partial_l\partial_{(i}\delta\gamma_{j)m}\right] + \mathcal{O}(\text{pert}^2)\ . \tag{3.35}$$

In 3 dimensions the Weyl tensor is identically zero, so the Riemann tensor can be expressed solely in terms of the Ricci tensor and the metric. Both a_i, R_{ij} and their derivatives are of the order of perturbations, so any potential term which is cubic in these will be of higher order in the quadratic truncation. This observation immediately reduces considerably the number of possible terms.

Even after restricting the terms to be quadratic in the acceleration, curvature and their derivatives, there are still several terms which are redundant at the level of the quadratic action around Minkowski. For instance, since the curvature is of the order of perturbations, we can further identify redundant terms by commuting the covariant derivatives, i.e. $\nabla_{[i}\nabla_{j]}(\text{pert}) = \mathcal{O}(\text{pert}^2)$. Moreover, performing integration by parts, some terms turn out to give the same contribution up to higher order terms in perturbative expansion, e.g. the term $N\nabla_i\mathsf{R}\,a^i$ can be written as $-N\mathsf{R}\nabla_i a^i$ up to a boundary term and $\mathsf{R}\,a_i a^i$ (which does not contribute at the level of our quadratic truncation). Finally, making use of the contracted Bianchi identities $\nabla^j\mathsf{R}_{ij} = \nabla_i\mathsf{R}/2$, we find that the only terms which contribute to the action at quadratic level in perturbations are

$$\begin{aligned}
\mathcal{L}_{z=1} &= 2\alpha\, a_i a^i + \beta\,\mathsf{R}\,, \\
\mathcal{L}_{z=2} &= \alpha_1\,\mathsf{R}\nabla_i a^i + \alpha_2\,\nabla_i a_j\nabla^i a^j + \beta_1\,\mathsf{R}_{ij}\mathsf{R}^{ij} + \beta_2\,\mathsf{R}^2\,, \\
\mathcal{L}_{z=3} &= \alpha_3\,\nabla^2\mathsf{R}\nabla_j a^j + \alpha_4\,\nabla^2 a_i\nabla^2 a^i + \beta_3\,\nabla_i\mathsf{R}_{jk}\nabla^i\mathsf{R}^{jk} + \beta_4\,\nabla_i\mathsf{R}\nabla^i\mathsf{R}\,,
\end{aligned} \tag{3.36}$$

where we defined $\nabla^2 \equiv \nabla_i\nabla^i$. This is the most general version of Hořava theory including all terms that contribute to linear perturbations around Minkowski background. Just as a remark, notice that the projectable version of the theory with $N = N(t)$ can be obtained simply by taking the limit $\alpha_i \to \infty$ (Blas et al., 2010b).

At this point we need to introduce the additional terms we wish to consider, which are the mixed 2-time and 2-space derivative terms. Apart from the generic form $(\nabla_i K_{jk})^2$ chosen in Pospelov and Shang (2012), one could also write terms of the form $(K_{ij}a_k)^2$ and $K_{ij}K^j_l \mathrm{R}^{il}$, by appropriate contractions with the metric γ_{ij}. However, considering the perturbations in (3.33), we find that

$$K_{ij} = \frac{1}{2}\left[\delta\dot{\gamma}_{ij} - \partial_i \delta N_j - \partial_j \delta N_i\right] + \mathscr{O}(\text{pert}^2) \ . \tag{3.37}$$

In other words, the extrinsic curvature is first order in perturbations; for this reason only the terms of the form $(\nabla_i K_{jk})^2$ will contribute to the quadratic action.

There is however one more possible combination that contributes at the same order of perturbation and that respects the $\mathrm{Diff}_{\mathcal{F}}$ symmetry. Such combination is given by the contraction of the time derivative of the acceleration and the shift vector (Coates et al., 2016). The action for the mixed derivatives part of the action can thus be written as

$$\begin{aligned}\mathcal{S}_\times = \frac{M_{pl}^2}{2M_*^2}\int \mathrm{d}t\,\mathrm{d}^3x\, N\sqrt{\gamma}\,\Big[M^{ijklmn}\nabla_i K_{jk}\nabla_l K_{mn} + \\ +2\left(\zeta_1 \mathcal{A}_i\mathcal{A}^i + \zeta_2\mathcal{A}_i\nabla^i K + \zeta_3\mathcal{A}_i\nabla_j K^{ij}\right)\Big] \ .\end{aligned} \tag{3.38}$$

In this expression, the tensor M^{ijklmn} is given by

$$M^{ijklmn} \equiv \sigma_1\gamma^{ij}\gamma^{lm}\gamma^{kn} + \sigma_2\gamma^{il}\gamma^{jm}\gamma^{kn} + \sigma_3\gamma^{il}\gamma^{jk}\gamma^{mn} + \sigma_4\gamma^{ij}\gamma^{kl}\gamma^{mn} \ , \tag{3.39}$$

while the vector $\mathcal{A}_i$ is given by

$$\mathcal{A}_i \equiv \frac{1}{2N}\left(\dot{a}_i - N^j\nabla_j a_i - a_j\nabla_i N^j\right) \ . \tag{3.40}$$

The term with coefficient σ_1 in (3.39) corresponds to the one introduced in Pospelov and Shang (2012), used there to remove the quadratic divergences in the vector loops.

Finally, we should point out that there is yet another combination, this time including time derivatives of the 3-dimensional curvature, which should in principle be included in the action. This combination though turns out to be redundant at the level of the quadratic action in perturbations around flat spacetime (see Appendix A of Coates et al., 2016) and hence we didn't incude them here.

3.4 PERTURBATIONS AROUND FLAT SPACETIME

We now want to study the perturbations around flat spacetime in the theory with mixed derivatives introduced in the previous Section.

Decomposing the perturbations with respect to their transformation properties under spatial rotations, the base quantities of the theory are expanded to first order of perturbation as

$$\begin{aligned} &N = 1 + A\,, \qquad N^i = (B^i + \partial^i B)\,, \\ &\gamma_{ij} = \delta_{ij}(1+2\psi) + \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\triangle\right) E + \partial_{(i}E_{j)} + h_{ij}\,, \end{aligned} \tag{3.41}$$

where $\partial_i B^i = \partial_i E^i = \delta^{ij} h_{ij} = \partial_i h^{ij} = 0$. We remark here that, since we are not employing the projectability condition, we allow $A = A(t, \boldsymbol{x})$.

In the gravity sector, there are 2 tensor degrees of freedom coming from h_{ij}, 4 vector degrees of freedom coming from B_i and E_i, and 4 scalar degrees of freedom given by A, B, E, ψ. The set of all these produces a total of 10 perturbations, which exhaust the number of degrees of freedom that can reside in a foliated 4-metric.

Additionally, in the following we will expand the perturbations into plane waves through

$$Q(t, \boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k\, Q_{\boldsymbol{k}}(t)\, e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}}\,, \tag{3.42}$$

where $Q(t, \boldsymbol{x})$ represents any perturbation and $Q_{\boldsymbol{k}}(t)$ is the corresponding mode function, satisfying the reality condition $Q_{-\boldsymbol{k}} = Q^\star_{\boldsymbol{k}}$. Thanks to the invariance of the Minkowski background under spatial rotations and the presence of a preferred foliation, the resulting quadratic action will depend only on the magnitude of the momentum $k \equiv |\boldsymbol{k}|$ and all sectors will decouple from the each other. In the remainder, we will omit the subscript $\boldsymbol{k}$ in the mode functions $Q_{\boldsymbol{k}}$ to limit the clutter in the formulas.

3.4.1 *Tensor sector*

Tensor modes are only affected by the first term in (3.38); the action quadratic in tensor perturbations is then given by

$$\mathcal{S}^{(2)}_{\text{tensor}} = \frac{M_p^2}{8} \int \mathrm{d}t\, \mathrm{d}^3k \left(1 + \sigma_2 \kappa^2\right) \left(|\dot{h}_{ij}|^2 - \omega_T^2 |h_{ij}|^2\right) \,, \tag{3.43}$$

where we defined $\kappa \equiv k/M_*$ for convenience. The dispersion relation for the tensor perturbations is given by

$$\omega_T^2 = k^2 \, \frac{\beta - \beta_1 \kappa^2 - \beta_3 \kappa^4}{1 + \sigma_2 \kappa^2} \,. \tag{3.44}$$

Linear stability of the tensor perturbations can be attained by requiring a positive kinetic term and a real frequency. In the UV, i.e. for $\kappa \gg 1$, the kinetic term is dominated by the κ^2 part, which imposes $\sigma_2 > 0$. The dispersion relation in this regime is

$$\omega_T^2 = -\frac{\beta_3 k^2}{\sigma_2} \left[\kappa^2 + \mathcal{O}(\kappa^0)\right] \,, \tag{3.45}$$

thus requiring $\beta_3/\sigma_2 < 0$.

In the IR, i.e. for $\kappa \ll 1$, the kinetic term is manifestly positive, so the only constraint comes from requiring a real propagation speed;

$$\omega_T^2 = \beta\, k^2 \left[1 + \mathcal{O}(\kappa^2)\right] \,. \tag{3.46}$$

Collecting all the conditions obtained above for the stability of tensor modes, we have

$$\sigma_2 > 0 \,, \qquad \beta_3 < 0 \,, \qquad \beta > 0 \,. \tag{3.47}$$

3.4.2 *Vector sector*

The original motivation for the mixed derivative extension of Hořava gravity was to overcome the technical naturalness problem in the suppression mechanism studied in Pospelov and Shang (2012). Although the four vector perturbations B_i and E_i correspond to two gauge modes and two non-dynamical modes, the gauge invariant combination $B_i - \dot{E}_i/2$ will still be generated virtually in graviton loops. However, in standard Hořava gravity, the vector propagator remains the same as in GR. As the suppression mechanism relies on loop integrals that are regulated in the UV, the vector loops lead to quadratic divergences. The addition of mixed derivative terms provides the necessary contribution to the vector propagator.

Considering that the quantity $\mathcal{A}_i$ in (3.38) contains only scalar perturbations, the vector sector is only affected by the first term in (3.38). The action quadratic in vector perturbations thus becomes

$$\mathcal{S}^{(2)}_{\text{vector}} = \frac{M_{pl}^2}{4} \int \mathrm{d}t\, \mathrm{d}^3k\, k^2 \left[1 + \frac{\kappa^2}{2}(\sigma_1 + 2\sigma_2)\right] \left|B^i - \frac{\dot{E}^i}{2}\right|^2 . \tag{3.48}$$

In coordinate space, the equation of motion for the non-dynamical mode B_i is given by

$$\left(1 - \frac{(\sigma_1 + 2\sigma_2)}{M_p^2} \triangle\right) \triangle \left(B^i - \frac{\dot{E}^i}{2}\right) = 0\,, \tag{3.49}$$

where $\triangle \equiv \delta^{ij}\partial_i\partial_j$ is the the flat-space Laplace operator. If we impose, as a boundary condition, that all perturbations and all their derivatives vanish asymptotically, then the unique solution is

$$B^i = \frac{1}{2}\dot{E}^i\,. \tag{3.50}$$

Replacing this solution back in the action, we find that the action vanishes up to boundary terms. Hence, there are no propagating vector modes. It is clear, however, that the σ_1 and σ_2 terms modify the behavior of the vector modes by introducing extra spatial derivatives and thus making the propagator decay as $1/k^4$ in the UV. This is precisely the feature that was used in Pospelov and Shang (2012) to remove the divergences related to the vector modes.

3.4.3 *Scalar sector*

We can now proceed to study the scalar sector of the theory, which is where the most interesting features lie. The quadratic action for this sector is

$$\begin{aligned}
\mathcal{S}^{(2)}_{\text{scalar}} = \frac{M_{pl}^2}{2}\int \mathrm{d}t\,\mathrm{d}^3k \Bigg\{ & \left[3\,(1-3\lambda)+(\sigma_1+3\sigma_2+9\sigma_3+3\sigma_4)\,\kappa^2\right]\left|\dot{\psi}+\frac{k^2}{6}\dot{E}\right|^2 \\
& +\frac{\zeta_1\kappa^2}{2}|\dot{A}|^2+k^2\left(2\alpha+\alpha_2\kappa^2+\alpha_4\kappa^4\right)|A|^2 \\
& +2k^2\left[\beta+(3\beta_1+8\beta_2)\,\kappa^2+(3\beta_3+8\beta_4)\,\kappa^4\right]\left|\psi+\frac{k^2}{6}E\right|^2 \\
& +k^4\left[1-\lambda+(\sigma_1+\sigma_2+\sigma_3+\sigma_4)\,\kappa^2\right]\left|B-\frac{\dot{E}}{2}\right|^2 \\
& +2k^2\left(\beta-\alpha_1\kappa^2+\alpha_3\kappa^4\right)\left[A^*\left(\psi+\frac{k^2}{6}E\right)+\text{c.c.}\right] \\
& +k^2\left[1-3\lambda+\tilde{\Sigma}\,\kappa^2\right]\left[\left(B-\frac{\dot{E}}{2}\right)^*\left(\dot{\psi}+\frac{k^2}{6}\dot{E}\right)+\text{c.c.}\right] \\
& +k^2\frac{\kappa^2\,(\zeta_2+\zeta_3)}{2}\left[\left(B-\frac{\dot{E}}{2}\right)^*\dot{A}+\text{c.c.}\right] \\
& +\frac{\kappa^2\,(3\zeta_2+\zeta_3)}{2}\left[\dot{A}^*\left(\dot{\psi}+\frac{k^2}{6}\dot{E}\right)+\text{c.c.}\right]\Bigg\}\,,
\end{aligned} \tag{3.51}$$

where "c.c." denotes the complex conjugate of the preceeding expression, and we have defined for convenience $\tilde{\Sigma} \equiv \sigma_1+\sigma_2+3\sigma_3+2\sigma_4$.

This action is manifestly gauge invariant as, at linear order, the quantities

$$\Psi \equiv \psi+\frac{k^2}{6}E\,, \quad \mathcal{B} \equiv B-\frac{1}{2}\dot{E}\,, \quad \text{and} \quad kA \tag{3.52}$$

are invariant under $\mathrm{Diff}_{\mathcal{F}}$. Note that the perturbation A is a scalar under 3D diffeomorphisms, but under time reparametrizations of the type $t \to t+f(t)$, it transforms as $A \to A+f'(t)$. Therefore, the quantity that is gauge invariant is $\partial_i A$ as opposed to A. As a consequence, the gauge invariant plane wave mode function is kA.

As we can see in the scalar action (3.51), we are left with three scalar degrees of freedom, two of which are dynamical. We can now use the momentum constraint to replace $\mathcal{B}$, obtaining

$$\mathcal{B} = -\frac{1}{k^2}\frac{1}{1-\lambda+(\sigma_1+\sigma_2+\sigma_3+\sigma_4)\,\kappa^2} \\ \times\left[\left(1-3\lambda+(\sigma_1+\sigma_2+3\sigma_3+2\sigma_4)\,\kappa^2\right)\left(\dot{\psi}+\frac{k^2}{6}\dot{E}\right)+\frac{\zeta_2+\zeta_3}{2}\kappa^2\dot{A}\right]. \quad (3.53)$$

Unlike the usual case in Hořava gravity, we see that this time the field A is dynamical; for this reason we cannot perform any further reductions. We then have a scalar action with two dynamical degrees of freedom, $Y=(\Psi, A)$, which can be written more conveniently as

$$\mathcal{S}^{(2)}_{\text{scalar}} = \frac{M_{pl}^2}{2}\int \mathrm{d}t\,\mathrm{d}^3k\,\left(\dot{Y}^\dagger\boldsymbol{\mathcal{K}}\dot{Y}-Y^\dagger\boldsymbol{\mathcal{M}}Y\right), \quad (3.54)$$

where the matrices $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\mathcal{M}}$ are symmetric 2×2 matrices. The kinetic matrix $\boldsymbol{\mathcal{K}}$ has components

$$\begin{aligned}
\boldsymbol{\mathcal{K}}_{11} &= 6+(4\sigma_1+6\sigma_2)\,\kappa^2+\frac{4+[8\,(\sigma_1+\sigma_2)+4\sigma_4]\,\kappa^2+[2\,(\sigma_1+\sigma_2)+\sigma_4]^2\,\kappa^4}{\lambda-1-(\sigma_1+\sigma_2+\sigma_3+\sigma_4)\,\kappa^2}\,, \\
\boldsymbol{\mathcal{K}}_{12} &= -\zeta_3\kappa^2-\frac{\zeta_2+\zeta_3}{2}\frac{2\kappa^2+[2\,(\sigma_1+\sigma_2)+\sigma_4]\,\kappa^4}{\lambda-1-(\sigma_1+\sigma_2+\sigma_3+\sigma_4)\,\kappa^2}\,, \\
\boldsymbol{\mathcal{K}}_{22} &= \frac{\zeta_1\kappa^2}{2}+\frac{(\zeta_2+\zeta_3)^2\,\kappa^4}{4\,[\lambda-1-(\sigma_1+\sigma_2+\sigma_3+\sigma_4)\,\kappa^2]}\,,
\end{aligned} \quad (3.55)$$

while the mass matrix $\boldsymbol{\mathcal{M}}$ has components

$$\begin{aligned}
\boldsymbol{\mathcal{M}}_{11} &= -2k^2\left[\beta+(3\beta_1+8\beta_2)\,\kappa^2+(3\beta_3+8\beta_4)\,\kappa^4\right], \\
\boldsymbol{\mathcal{M}}_{12} &= -2k^2\left[\beta-\alpha_1\kappa^2+\alpha_3\kappa^4\right], \\
\boldsymbol{\mathcal{M}}_{22} &= -k^2\left[2\alpha+\alpha_2\kappa^2+\alpha_4\kappa^4\right].
\end{aligned} \quad (3.56)$$

The non-diagonal kinetic matrix can be diagonalized by performing a rotation to a new field basis $\boldsymbol{\mathcal{Z}}$ through

$$\boldsymbol{\mathcal{Z}} \equiv R^{-1}Y\,, \quad (3.57)$$

with the rotation

$$R = \begin{pmatrix} 1 & -\frac{\boldsymbol{\mathcal{K}}_{12}}{\boldsymbol{\mathcal{K}}_{11}} \\ 0 & 1 \end{pmatrix}. \quad (3.58)$$

In the new field basis, the kinetic matrix is diagonal $R^T \boldsymbol{\mathcal{K}}\, R = \mathrm{diag}(\overline{\mathcal{K}}_1\,, \overline{\mathcal{K}}_2)$ with eigenvalues

$$\overline{\mathcal{K}}_1 = \boldsymbol{\mathcal{K}}_{11}\,, \quad \overline{\mathcal{K}}_2 = \frac{\det \boldsymbol{\mathcal{K}}}{\boldsymbol{\mathcal{K}}_{11}}\,. \tag{3.59}$$

It should be noted that this procedure is not unique. For instance, one could choose $\boldsymbol{\mathcal{K}}_{22}$ and $(\det\boldsymbol{\mathcal{K}}/\boldsymbol{\mathcal{K}}_{22})$ for the kinetic eigenvalues, or adopt a basis obtained through an orthogonal rotation. However, the latter produces very complicated eigenvalues, rendering the treatment much more inconvenient. Provided that the rotation has non-zero determinant (i.e. the transformation can be inverted), the stability conditions are compatible.

The first eigenvalue in (3.59) is independent of ζ_1, ζ_2, ζ_3, while the second one vanishes when these parameters are zero. Hence, we identify the former mode as the scalar graviton of standard Hořava theory. In the IR the eigenvalues (3.59) reduce to

$$\overline{\mathcal{K}}_1 = \frac{2\,(3\lambda-1)}{\lambda-1} + \mathcal{O}(\kappa^2), \quad \overline{\mathcal{K}}_2 = \frac{\zeta_1 \kappa^2}{2} + \mathcal{O}(\kappa^4)\,, \tag{3.60}$$

leading to the following conditions for avoiding a ghost instability

$$\frac{3\lambda-1}{\lambda-1} > 0\,, \quad \zeta_1 > 0\,. \tag{3.61}$$

Thanks to the large number of UV relevant operators, there is more freedom to avoid high energy ghosts. In the $\kappa \gg 1$ limit, the kinetic eigenvalues become

$$\overline{\mathcal{K}}_1 = \left[2\,(\sigma_2+2\sigma_3) - \frac{(2\sigma_3+\sigma_4)^2}{\sigma_1+\sigma_2+\sigma_3+\sigma_4}\right]\kappa^2 + \mathcal{O}\left(\kappa^0\right)\,, \tag{3.62}$$

$$\overline{\mathcal{K}}_2 = \left[\frac{\zeta_1}{2} - \frac{(2\sigma_1+3\sigma_2)\,\zeta_2^2 + 2\,(\sigma_2-\sigma_4)\,\zeta_2\zeta_3 + (\sigma_2+2\sigma_3)\,\zeta_3^2}{4\sigma_1\,(\sigma_2+2\sigma_3) + 4\sigma_2\,(\sigma_2+3\sigma_3) + 4\sigma_2\sigma_4 - 2\sigma_4^2}\right]\kappa^2 + \mathcal{O}\left(\kappa^0\right)\,.$$

Finally, we can obtain the dispersion relations. The equation of motion for the mode functions Y can be obtained by varying the reduced action (3.54) with respect to $Y^\dagger$

$$\boldsymbol{\mathcal{K}}\,\ddot{Y} + \boldsymbol{\mathcal{M}}\,Y = 0\,. \tag{3.63}$$

We can then easily find the eigenfrequencies by considering a mode with $Y = Y_0 e^{-i\omega t}$ and solving the equation

$$\det\left[(-i\omega)^2 \boldsymbol{\mathcal{K}} + \boldsymbol{\mathcal{M}}\right] = 0\,, \tag{3.64}$$

which gives two distinct solutions for ω^2. We will avoid reporting here the exact form of the dipersion relations, as they are not particularly elegant and do not provide enough interesting information to justify their presence. For the scopes of the present discussion, it will suffice to have the expressions for the dispersion relations in the IR; these are given by

$$\frac{\omega_1^2}{M_*^2} = \frac{\beta\,(\beta-\alpha)\,(\lambda-1)}{\alpha\,(3\lambda-1)}\kappa^2 + \mathcal{O}\left(\kappa^4\right)\,, \tag{3.65}$$
$$\frac{\omega_2^2}{M_*^2} = -\,\frac{4\alpha}{\zeta_1} + \left[\frac{2\,[\beta\zeta_1 - \alpha\,(3\zeta_2+\zeta_3)]^2}{\alpha\zeta_1^2\,(3\lambda-1)} - \frac{(\beta\zeta_1+2\alpha\zeta_3)^2}{\alpha\zeta_1^2} - \frac{6\alpha_2}{\zeta_1}\right]\frac{\kappa^2}{3} + \mathcal{O}\left(\kappa^4\right)\,.$$

We remark that the first expression has exactly the form of the IR dispersion relation for the scalar graviton in standard Hořava gravity, which upon imposing the stability of tensor modes (3.47) and positivity of the kinetic terms (3.61), retains the familiar condition

$$\beta > \alpha > 0\,, \tag{3.66}$$

to have a real propagation speed. On the other hand, the second mode has a tachyonic instability at leading order. It can in fact be seen easily from the conditions given in (3.61) and (3.66) that the second scalar mode has a negative squared mass.

The presence of this last mode, which develops a tachyonic instability, is quite worrisome. Its presence could in fact hinder the consistency of the extended theory we are analysing, and it throws a shadow of doubt on whether adding mixed derivative terms to Hořava gravity is a good way to get rid of the quadratic divergences in the vector sector. Before becoming too pessimistic though, it might be interesting to find out more about the ill-behaved scalar model.

3.5 THE SCALAR SECTOR IN THE IR LIMIT

One might be tempted to assume that the higher dimensional mixed derivative operators (3.38) are UV deformations, irrelevant from the perspective of the low-energy effective theory. However, from (3.65) we see that at leading order, the dispersion relation of the second mode in the IR depends on the coupling constant ζ_1 from a mixed derivative term. This is because the term $\left(\mathcal{A}_i\mathcal{A}^i\right)$ actually generates a kinetic term for an otherwise non-propagating perturbation in standard Hořava gravity. In that regard, the mixed derivative term $\left(\mathcal{A}_i\mathcal{A}^i\right)$ is an IR relevant term as it provides the low-energy kinetic term for the, now dynamical, lapse perturbation A. However, due to the two additional spatial derivatives in this term, the would-be gradient term $a_i a^i$ now provides a mass to A.

It is therefore instructive to consider the IR theory and present a cleaner and more concise re-derivation of the perturbative dynamics. This will clearly describe the source of the new degree of freedom. We drop all the UV relevant terms such that the resulting action preserves the number of degrees of freedom of the full theory, obtaining

$$\mathcal{S}_{\text{IR}} = \frac{M_p^2}{2}\int \mathrm{d}t\,\mathrm{d}^3x\,N\sqrt{\gamma}\left[K_{ij}K^{ij} - \lambda K^2 + 2\alpha a_i a^i + \beta R + \frac{2}{M_*^2}\zeta_1\mathcal{A}_i\mathcal{A}^i\right]. \tag{3.67}$$

As we are interested only in the scalar sector of the theory, we fix the gauge and decompose the metric degrees of freedom as[3]

$$N = 1 + A\,, \quad N^i = \partial^i B\,, \quad \gamma_{ij} = \delta_{ij}\left(1 + 2\psi\right)\,. \tag{3.68}$$

Expanding the action up to quadratic order in perturbations, we arrive at the action

$$\mathcal{S}^{(2)}_{\text{IR,scalar}} = \frac{M_p^2}{2}\int \mathrm{d}t\,\mathrm{d}^3x\,\mathcal{L}_{\text{IR}}\,, \tag{3.69}$$

3 Notice that, as before, A and ψ will represent the dynamical fields, while B will turn out to be non-dynamical and will therefore be integrated out.

with

$$\mathcal{L}_{\text{IR}} = -3\,(3\lambda - 1)\,\dot{\psi}^2 + \frac{\zeta_1}{2M_*^2}\nabla_i \dot{A}\nabla^i \dot{A} + 2\,(3\lambda - 1)\,\triangle B\,\dot{\psi} + 2\beta\nabla_i\psi\nabla^i\psi \\ + 2\alpha\nabla_i A\nabla^i A + 4\beta\nabla_i A\nabla^i\psi - (\lambda - 1)\,(\triangle B)^2\ . \quad (3.70)$$

Integrating out the non-dynamical mode B, the reduced action becomes

$$\mathcal{L}_{\text{IR}} = \frac{2\,(3\lambda - 1)}{\lambda - 1}\dot{\psi}^2 + \frac{\zeta_1}{2M_*^2}\nabla_i \dot{A}\nabla^i \dot{A} \\ + 2\alpha\nabla_i A\nabla^i A + 4\beta\nabla_i A\nabla^i\psi + 2\beta\nabla_i\psi\nabla^i\psi\ . \quad (3.71)$$

Due to the lack of kinetic mixing between A and ψ we can immediately read off the no-ghost conditions, given by

$$\frac{3\lambda - 1}{\lambda - 1} > 0\,, \quad \text{and} \quad \zeta_1 > 0\,, \quad (3.72)$$

as before. Furthermore, as the canonically normalized field is $\nabla_i A$, the leading order contribution to the dispersion relation of this field comes from the second and third terms in the above action, allowing us to read off the mass of the massive mode as

$$m^2 = -\frac{4M_*^2\alpha}{\zeta_1}\ . \quad (3.73)$$

This exercise demonstrates then that at leading order the unstable degree of freedom corresponds to the gradient of the lapse, i.e. $\nabla_i A$ which acquires a negative squared mass. The other scalar degree of freedom is massless and corresponds to the Hořava scalar.

3.5.1 *Changing the nature of the instability*

We have found above that the new scalar degree of freedom has a tachyonic instability, provided that the stability conditions for the other modes, given in (3.47), (3.61) and (3.66), are satisfied. On the other hand, by relaxing one of these conditions, it is possible to obtain a real mass for the new degree of freedom. There are three ways to accomplish this:

i. for $\alpha < 0 < \beta$ the first scalar mode has a gradient instability;

ii. for $\beta < \alpha < 0$ the tensor mode becomes a ghost;

iii. for $\zeta_1 < 0$ the second scalar mode is a ghost.

The limits on the parameters of the Hořava scalar and the tensor modes are well established (Blas et al., 2011; Yagi et al., 2014; Audren et al., 2015), so we choose to preserve the stability conditions for the modes already present in the standard Hořava theory. This leaves us with the third option. In fact, if we allow the IR effective theory to have a ghost with a mass larger than the cutoff of the low-energy action (strong coupling scale) M_{sc} (Papazoglou and Sotiriou, 2010; Kimpton and Padilla, 2010) then the ghost will not be generated in the regime of validity of the effective field theory (Blas et al., 2011). This is an approach frequently used in effective field theories. However, here we actually know the UV completion of the theory, so we can eventually verify if the UV terms do indeed exorcise the ghost.

For the IR effective theory to stay weakly coupled at all relevant scales one needs $M_* < M_{sc}$. This choice ensures that the higher derivative terms in the action become relevant before the IR theory becomes strongly coupled (Blas et al., 2010a). Then, the conditions for having a heavy ghost *and* for avoiding strong coupling can be combined into

$$\frac{4\,\alpha}{|\zeta_1|} > \frac{M_{sc}^2}{M_*^2} > 1\,, \tag{3.74}$$

where we assumed $\zeta_1 < 0$. For the present discussion, we will assume $|\zeta_1| \ll \alpha$, which is necessary but not sufficient for satisfying the above conditions, although the details of our argument will not change in the case of a larger hierarchy between M_{sc} and M_*.

From the analysis in the previous Section, it is clear that the ghost degree of freedom is not an artifact of some truncation (as is the usual assumption in effective field theories that contain a very massive ghost) but it actually continues to exist and propagate in the UV theory. Hence, the only way to have positive energy at high momenta is if the kinetic term for this scalar changes sign at some intermediate momentum. On the other hand, in the deep IR, the equation of motion for the new degree is, up to boundary conditions,

$$-\frac{|\zeta_1|}{2\,M_*^2}\ddot{A}-\alpha\,A=0\,. \tag{3.75}$$

The coefficient of the kinetic term and the mass term have the same sign for positive α and before a canonical normalization. This suggests that when the former changes sign the latter should as well, else the scalar mode will turn from being a ghost to being classically unstable.

Clearly one needs to go beyond the IR limit of the dispersion relation in order to get the full picture. To make this discussion concrete, we chose an example parameter set which is compatible with the current bounds on the IR parameters, given by

$$\begin{aligned}
&\alpha=10^{-7}\,, \qquad \beta-1=1.5\times10^{-7}\,, \qquad \lambda-1=10^{-8}\,,\\
&\alpha_1=\alpha_2=\beta_1=\beta_2=-1\,, \qquad \alpha_3=\alpha_4=\beta_3=\beta_4=-2\,,\\
&\sigma_1=\sigma_2=\sigma_3=1\,,\ \sigma_4=-13\,,\ \zeta_1=-10^{-8}\,,\ \zeta_2=\zeta_3=1\,.
\end{aligned} \tag{3.76}$$

With these parameters, the standard Hořava scalar is stable both in the IR and UV, while the new mode is a heavy ghost in the IR and stable in the UV. In Fig. 2, we show the kinetic terms for each mode as a function of momenta. The second mode is the new degree of freedom. Notice that at around $k\simeq 10^{-4}M_*$, the sign of the kinetic term flips, and the mode becomes healthy again. This is due to the second term in $\overline{\mathcal{K}}_2$ (3.62) becoming dominant. In Fig. 3, we show the dispersion relation as a function of the momentum. The first mode, i.e. the scalar graviton of Hořava theory has a dispersion relation $\propto k^2$ in the IR and $\propto k^4$ in the UV, as expected. The second mode starts off with a constant mass ($> M_*$), but when its kinetic term crosses zero and flips its sign the frequency of the mode diverges. It then experiences a tachyonic instability between momenta $10^{-4}M_* < k < M_*$. This implies that the theory is actually unstable at low-energies and the IR truncation that we used earlier to argue that the new scalar is a heavy ghost in the IR is simply misleading.

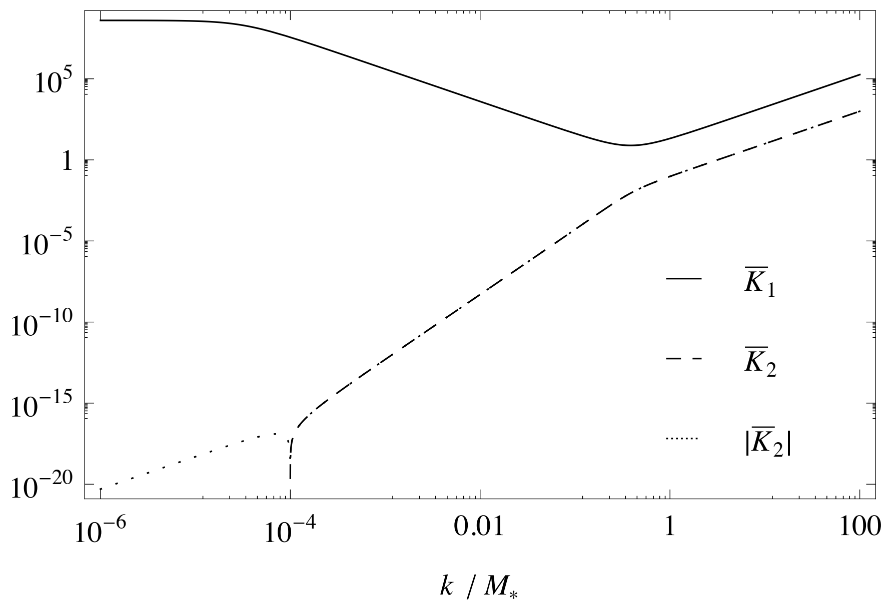


Figure 2: The kinetic matrix eigenvalues (3.59) for the parameter set (3.76). The first eigenvalue (solid line) corresponds to the scalar graviton of Hořava gravity, while the second eigenvalue (dashed line, with absolute value shown as dotted line) corresponds to the new degree arising from the mixed derivative extension. With the chosen parameters, the latter ceases being a ghost at momenta $k \simeq 10^{-4} M_*$.

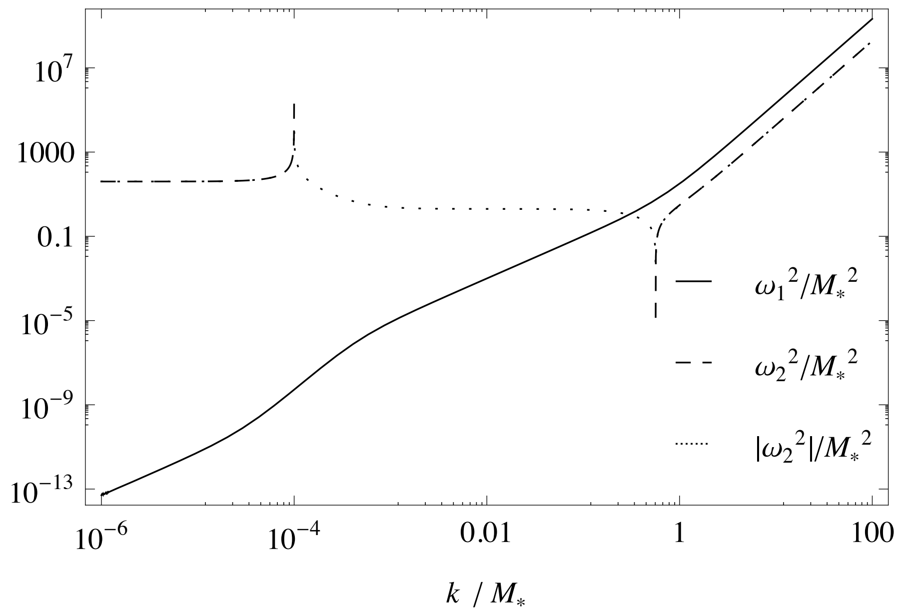


Figure 3: The dispersion relation of the two modes, analytically obtained by solving (3.64) then evaluated using the parameter set (3.76). The solid line corresponds to the first (Hořava) mode, the dashed line corresponds to the new scalar. The dotted line is the absolute value of the frequency of the second mode.

It seems likely that one could actually fine-tune the parameters of the theory so as to make the sign flip in the kinetic term exactly coincide with the one in the frequency and avoid any instability at any momenta. The complexity of the full dispersion relations in the diagonal basis makes it particularly challenging to find such a tuning in practice. However, it is hard to imagine how it would be radiatively stable even if it did exist.

3.6 possible solutions to the conundrum

As we have seen in the previous Sections, our hopes for the mixed derivative terms added to Hořava gravity in Pospelov and Shang (2012) to be able to solve the naturalness problem in the scalar sector could be disappointed. In Section 3.4.3 we have seen how the new terms create a tachyonic instability; such instability can be traded for a ghost, but the theory doesn't become healthy. Also, the possibility of this instability to be simply an artifact of the truncation of the high energy operators — as typically happens in Effective Field Theories — is not there, since for once we do know the UV completion of the theory and the scalar mode that is ill-behaved in the IR does propagate to the UV as well.

As a side note, it may be interesting to point out here that the presence of the second (problematic) scalar degree of freedom was recently confirmed in Klusoň (2016) through the Hamiltonian analysis of the theory we propose. In this work it was pointed out that the dangerous scalar degree of freedom could be eliminated by introducing some ad hoc Lagrange multiplier able to get rid of the unwanted degree of freedom; on the other hand this would introduce second order constraints in the theory thus rendering the theory itself extremely complicated.

If we have any hope left at all to be able to resolve the naturalness problem uncovered in Pospelov and Shang (2012), we will have to find a way to avoid the presence of the problematic scalar mode. In the following we will therefore try to discuss briefly two possible simple solutions to the problem.

3.6.1 *Fine tuning*

What we would ultimately wish to obtain here is to eliminate from the action (3.38) the dangerous terms, represented by the ones that contain the time derivative of the lapse. We would like on the other hand to be able to keep the other mixed derivative terms, since they are the ones that allow to control the quadratic divergence in the vector sector. To put this in a slightly different way, we want to keep only the terms that can modify the vector dipersion relations, but we have no interest in including terms that can modify the scalar dispersion relation, since the latter is perfectly well behaved already in the usual formulation of Hořava gravity.

One first way that comes to mind in order to do so, is to choose a tuning of the couplings such that the ζ_i couplings can be disregarded and set to zero. By doing this, we fall back to the case that was studied in Colombo et al. (2015a). First of all in such case the vector modes, while remaining non-dynamical, acquire higher order derivatives; this feature is precisely what helps in eliminating the quadratic divergences associated to the vector graviton loops. In addition, for what concerns both the tensor and scalar sectors, the dispersion relations become $\omega^2 \propto k^4$ (Colombo et al., 2015a). This could in principle be a problem, since the renormalisability of Hořava gravity hinges on the fact that the dispersion relations for the gravitons behave as $\omega^2 \propto k^6$. On the other hand, as we have seen previously in Section 3.2, introducing mixed derivatives in fact modifies the anisotropic scaling indices in such a way that a theory can be renormalisable and unitary even when the dispersion relations are different form the usual Hořava ones (Colombo et al., 2015b). In particular for a theory in $3+1$-dimensions with $z = 3$ and mixed derivatives with only two time and two spatial derivatives (i.e. $y = 1$), the anisotropic scaling changes to $m = 2$ in (3.4) and the dispersion relations change to fourth order ones, while retaining renormalisability and unitarity. This is exactly the case of the mixed derivative extension (with ζ_i tuned to zero) considered in Pospelov and Shang (2012) and generalised in our work, so we can safely assume that this kind of theory would be well behaved.

The tuning described here is a completely arbitrary choice. We have no physical reason — other than convenience — to disregard the derivative of the lapse. Additionally the fact that such terms are of the same order as the ones that we introduce in the action means that they could still be generated by radiative corrections. For this reason, while this tuning could seemingly help in getting rid of the instabilities in the scalar sector of the extended theory, we would like to be able to find a different way to solve the problem.

3.6.2 *Invoking the projectability condition*

There is a more consistent way to obtain what we are after. As we pointed out already, the issues associated with the unstable extra degree stem from the terms with coefficients ζ_n, i.e. those that contain time derivatives of the acceleration vector thus making the lapse dynamical. On the other hand, the projectability condition (Hořava, 2009b) constrains the lapse to be a function of time only. Hence, if this condition were to be imposed, the offending terms would trivially vanish. In this restricted theory the lapse can be fixed by using the (space-independent) time reparametrization symmetry. We remark that projectable Hořava gravity (Hořava, 2009b; Sotiriou et al., 2009b,a; Weinfurtner et al., 2010) has recently been shown to be renormalizable (Barvinsky et al., 2016).

Imposing projectability affects only the scalar sector and the results obtained throughout this Chapter will remain the same for the tensor and vector modes. Thus, the stability conditions for the tensor modes are still given by (3.47) and the vector modes still acquire contributions from mixed derivative terms that improve the UV behavior.

The effect on the scalar sector is far more dramatic, as the projectability condition actually removes the second scalar mode. The coefficient of the kinetic term for the remaining scalar graviton is

$$\mathcal{K}_s = 6+(4\,\sigma_1+6\sigma_2)\,\kappa^2+\frac{4+[8\,(\sigma_1+\sigma_2)+4\sigma_4]\,\kappa^2+[2\,(\sigma_1+\sigma_2)+\sigma_4]^2\,\kappa^4}{\lambda-1-(\sigma_1+\sigma_2+\sigma_3+\sigma_4)\,\kappa^2}\,, \tag{3.77}$$

while the dispersion relation is given by

$$\omega_s^2 = \frac{-2\,\kappa^2\left[\beta+(3\,\beta_1+8\,\beta_2)\kappa^2+(3\,\beta_3+8\,\beta_4)\kappa^4\right]}{\mathcal{K}_s}\,. \tag{3.78}$$

In the UV, the dispersion relation becomes $\omega_s^2 \propto \kappa^4$, as expected from the modified scaling (3.4). In the opposite limit, the IR expression for the coefficient of the kinetic term yields

$$\mathcal{K}_s^2 = \frac{2\,(3\,\lambda-1)}{\lambda-1}\left[1+\mathscr{O}(\kappa^2)\right]\,, \tag{3.79}$$

while the dispersion relation reduces to

$$\omega_s^2 = -\frac{\beta\,(\lambda-1)}{3\,\lambda-1}\,k^2\left[1+\mathscr{O}(\kappa^2)\right]\,. \tag{3.80}$$

Requiring positivity of the kinetic term's coefficient (3.79) in this limit yields

$$\frac{3\,\lambda-1}{\lambda-1} > 0\,. \tag{3.81}$$

Combining the above with the conditions from the tensor sector (3.47), we see that the sound speed for the scalar mode is imaginary, leading to a gradient type instability.[4] This is in fact the well known gradient instability that plagues the projectable version of Hořava gravity, accompanied with strong coupling in the limit $\lambda \to 1$, which was discussed briefly in Section 2.2.2 of Chapter 2.

This solution therefore seems to introduce a different problem. On the other hand, when the mixed derivative extension of projectable Hořava gravity is taken in account, the scalar sector gets modified, and one could hope

4 In a cosmological setup, the amount of time necessary for the gradient instability to develop can be longer than the time scale of the Jeans instability, necessary for structure formation Mukohyama (2010).

that such modification could help in improving the behaviour of the theory. More work is required in order to establish if such extension of projectable Hořava theory is indeed well-behaved, as opposed to the usual projectable theory, or if the problems typically encountered within this theory do remain.

♦

The problem of how to couple a Lorentz violating sector to a Lorentz invariant matter sector is a fundamental one. Various proposals for a mechanism aimed at eliminating the percolation of Lorentz violating operators to the matter sector have been proposed. None of them has more merit than the others and all of them need more work before they can be accepted as a definite solution.

On the other hand, a further problem emerged within Hořava gravity, while studying one of such mechanisms. The vector sector presents a quadratic divergence that introduces a naturalness problem in the theory. A possible solution to such naturalness problem, consisting in the introduction of mixed derivatives to the basic version of the theory, was studied in the present Chapter. Our results show that this solution is not viable, because the scalar sector presents a tachyonic instability as a result of the new terms introduced in the theory. Some possible ways around this last problem were discussed, but they are more ad hoc solutions than general ones, and they are not particularly convincing.

More effort is therefore required in order to give a solution once and for all to the naturalness problem introduced by the vector sector of the theory.

4

CAUSAL STRUCTURE OF FOLIATED SPACETIMES

As we mentioned in the Introduction, in this thesis we try to address two distinct problems. The first problem, that of coupling Lorentz violating gravity to the matter sector, has been discussed in the previous Chapter.

The second problem typically encountered when considering Lorentz violating theories is that of the existence of black holes. As we mentioned previously, the very existence of black holes in general relativity hinges on the causal structure associated with Lorentz symmetry. As soon as Lorentz symmetry is broken it's hard to tell if black holes will still exist in the first place, and in case they do we will still need to find a good definition for them and their horizons.

In this and the next Chapters we want to address precisely this problem. In fact, black holes in the theories we have been considering so far have already been shown to exist (Eling and Jacobson, 2006; Barausse et al., 2011; Blas and Sibiryakov, 2011; Barausse and Sotiriou, 2013a; Sotiriou et al., 2014) and the event horizon for this type of black holes, dubbed the "universal horizon" (Barausse et al., 2011; Blas and Sibiryakov, 2011), seems to be a quite generic feature of Lorentz violating theories.

On the other hand, a rigorous definition of black holes in Lorentz violating theories is still missing. In the works listed above, black hole solutions were obtained in restricted settings, mainly spherical symmetry or slow rotation, and often by means of numerical integration of the equations of motion of the theory; analytic solutions exist but were also studied in restricted settings (Bhattacharyya and Mattingly, 2014). No real understanding of the casusal structure of such theories is yet available though, and a rigorous definition of horizon has not been found. This is the gap we tried to fill: in Bhattacharyya et al. (2016b) we analysed the type of causal structure emerging from theories that feature a preferred foliation, without necessarily referring to any particular theory, and we tried to identify a rigorous definition of horizon that doesn't depend on the symmetries of the solutions. Since this alone would be of little use, being a global definition, we also tried to work out a local characterisation of horizon and we studied the necessary conditions for a universal horizon to emerge in realistic situations.

Before starting, let's take a few lines to explain why we are interested in spacetimes with a preferred foliation. As we mentioned several times before, we are trying to have some more insight in some particular Lorentz violating gravity theories, which in the best case scenario might represent a renormalisable UV completion of standard general relativity. Having therefore Hořava gravity as the principal candidate in mind, it is natural to resort to the study of spacetimes with a preferred foliation.

In addition, the presence of a foliation that is for some reason preferred will introduce a preferred direction of time — usually defined by the normal vector to the leaves of such foliation. This breaks Lorentz symmetry in a quite obvious fashion. For this reason, it would be interesting to study the causal structure of such spacetimes and the possibility of the presence of black holes; this is in fact an interesting endeavour in its own right.

On the other hand, it turns out that this types of spacetime are actually the ones where the theories we mentioned above live. As we saw in Chapter 2, Hořava gravity is most easily formulated in a foliated spacetime, since the presence of the foliation makes straightforward to add only higher order spatial derivative operators to the theory. However, as we discussed in Section 2.2.3 of Chapter 2 (see also Jacobson, 2010), one can formulate the low-energy version of the theory in a covariant manner. It then becomes a generally covariant scalar-tensor theory where the scalar field (sometimes called the *khronon*) always has a timelike gradient everywhere, so that its level sets foliate the spacetime with spacelike hypersurfaces. These hypersurfaces impose a preferred notion of simultaneity. Indeed, the field equations become second order in time derivatives only in this preferred foliation (Blas et al., 2009; Jacobson, 2010; Kimpton and Padilla, 2013).

Additionally the theory contains an elliptic (instantaneous) mode (Blas and Sibiryakov, 2011) that implies instantaneous propagation of signals even at low energies. For this reason, since the instantaneous signals propagate along the leaves of the foliation, the foliation itself is what defines the causality in this spacetime.

We are now ready to proceed with the study of the basic aspects of causal structure in a spacetime with a preferred foliation. Even though we heavily draw intuition from Hořava gravity, we will adopt a more general viewpoint and we will try as much as possible to make no explicit reference to any field equations or actions. Additionally, we will never enter the specifics of how the preferred foliation comes about in the theory in question. The causal structure associated with a manifold can in fact be fully established on kinematical, topological, and geometrical considerations alone and our only assumption about the dynamics is that the theory in question has a well-posed initial value problem. Hence, our techniques and conclusions are in principle applicable to a broader class of theories than just Hořava gravity.

4.1 MANIFOLDS WITH A PREFERRED SPACELIKE FOLIATION

The spaces we wish to consider are described by the triplet $(\mathscr{M}, \Sigma, g)$, where $\mathscr{M}$ (the 'spacetime') is a *Hausdorff, paracompact, smooth, connected* and *foliated* manifold without boundary, Σ is the associated foliation structure, each leaf of which is a *connected, spacelike hypersurface* in $\mathscr{M}$, and g_{ab} is a Lorentzian metric on $\mathscr{M}$. Being a submanifold of $\mathscr{M}$, every leaf in the foliation is automatically Hausdorff, paracompact and smooth, although connectedness is not guaranteed. We will then impose that the leaves themselves be connected, as an additional assumption on physical grounds.

Owing to their spacelike nature, every leaf of the foliation represents a set of events which are *simultaneous* in an *absolute sense*, giving a pre-relativistic flavor to the physics that takes place on such spacetimes. A more operational understanding of this fact can be found while thinking about the physical situation we have in mind. In fact, the type of theories we are inspired by will admit propagation of signals at speed faster than light. Such signals will then propagate along a "lightcone" wider than the usual one. Increasing the speed of such signals, the lightcones will become wider and wider, until they effectively become the leaves of the foliation. Since this happens in the limit of infinite speed, we see that the events of one leaf are precisely the events where any signal coming from other parts of the same leaf are perceived as simultaneous.

Being such 'surfaces of simultaneity', the leaves can thus never intersect one another: indeed if they could there would be a breakdown of causality at the events where the intersection takes place. The geometrical property of a well-behaved foliation structure which automatically guarantees such elementary yet crucial requirements of causality is that *the foliation is ordered*. In particular, the foliation Σ associated with the triplet $(\mathscr{M}, \Sigma, \mathrm{g})$, by virtue of being *ordered* ensures that *every pair of distinct events in $\mathscr{M}$ will have a unique causal relation*. We will see this explicitly in the following section, after we propose a consistent definition of past and future compatible with the current notion of preferred simultaneity. However, we may already discuss some of these issues in an intuitive manner by appealing to the ordered nature of Σ.

The spacetime is everywhere foliated by assumption, so every event must reside on *at least one* leaf of Σ. Also, since the leaves must not intersect, every event resides on *at most one* leaf. Taken together, these two statements imply that every event in $\mathscr{M}$ will lie on a unique leaf of Σ. We may thus unambiguously denote a leaf of Σ by Σ_p if it contains the event p. By the same token, if the event $q \neq p$ is also contained in Σ_p, then $\Sigma_q = \Sigma_p$. Clearly, a leaf acts as a surface of simultaneity and it should not seem surprising that the existence of a foliation implies a suitable *causality condition*. We will see in the following that such causality condition will become the usual notion of *stable causality*.

In fact, at least in any 'sufficiently small' region of spacetime, one should be able to assign a unique real number T to each leaf in that region, such that the number varies from one leaf to the next in a *monotonic* manner preserving the ordering of the foliation. More formally one may always introduce a real monotonic function $\mathfrak{time}$

$$\mathfrak{time} : \Sigma \to \mathbb{R} \,, \qquad \mathfrak{time}(\Sigma_p) = T \in \mathbb{R} \,, \tag{4.1}$$

in any 'sufficiently small' region of spacetime, such that the set of all events with a given value of T represents the leaf on which the said events reside, i.e.

$$\Sigma_T \equiv \{q \in \mathscr{M} \mid \mathfrak{time}(q) = \mathfrak{time}(\Sigma_p) = T\} = \Sigma_p \,. \tag{4.2}$$

Furthermore, $\mathfrak{time}$ can be chosen to 'increase towards future' so that the leaf $\Sigma_{T'}$ is to the future of Σ_T if $T' > T$. A function $\mathfrak{time}$ satisfying the above criteria provides us with a *faithful time-parametrization of the foliation in the said region of spacetime*. From here onwards, we will adhere to common practice and denote a given choice of the $\mathfrak{time}$ function as well as its value at some event p by the same letter T.

The ordered nature of the foliation guarantees *at least one* faithful time-parametrization of the foliation in any 'sufficiently small' region of spacetime (but not necessarily globally). Whether or not this time-parametrization is unique will depend on the dynamical theory one has in mind. One could consider a theory that is invariant under the time-reparametrization of the foliation[1]

$$T \mapsto \tilde{T} = \tilde{T}(T) \ . \tag{4.3}$$

When such time-reparametrization is possible, the foliation is said to be ordered but not labeled. Hořava gravity is a characteristic example of a theory that enjoys symmetry under such time-reparametrizations. In the existing literature of Hořava gravity (see for example Blas et al., 2011), T is known as the *khronon* field and the leaves of the foliation are accordingly called constant khronon hypersurfaces. Clearly, one can also have theories that are invariant under limited time-reparametrizations, e.g. only time shifts or $T \to -T$.

In a theory where there is a uniquely labeled foliation, a breakdown of the preferred time-parametrization would necessarily signal a breakdown of the foliation structure itself. As we will see later on, it is rather common for a time-parametrization to break down and fail to provide a full cover of the manifold. Hence, restricting ones attention to theories with preferred time-parametrization or attempting to formulate causality relations by making reference to any specific time-parametrization is not advisable. Rather, it is best to make no reference to any labelling of the foliation leaves when discussing causality. Any restrictions coming from the existence of a preferred labelling could always be imposed *a posteriori* if needed.

One way to proceed would then be to employ more abstract tools from the theory of *totally ordered sets*. Yet, especially from a physics perspective, a formulation of causality in terms of curves that connect events and allow one to assign causal relationships between them — closer in spirit with general relativity— is perhaps preferable and desirable for multiple reasons:

1 By assumption, T furnishes a faithful time-parametrization here, although $\tilde{T}$ may not. In fact, $\tilde{T}$ will furnish a faithful time-parametrization as long as $(\mathrm{d}\tilde{T}/\mathrm{d}T) > 0$. If $(\mathrm{d}\tilde{T}/\mathrm{d}T) < 0$, one can regard $(-\tilde{T})$ as time-parameter which increases towards future. The faithfulness of the time-parametrization via $\tilde{T}$ breaks down where $(\mathrm{d}\tilde{T}/\mathrm{d}T) = 0$.

(i) first, one would be able to readily compare and contrast the present framework of causality with that of general relativity, underscoring this way the essential differences between them.

(ii) More importantly, the curves that allow one to establish causal relationship also model the propagation of signals. In particular, it would be natural to visualize causal development as a flow of data from one surface of simultaneity to the next along such curves. Therefore, this formalism seems to have a more direct bearing on the questions of determinism and predictability.

(iii) Last but not least, such a formalism is automatically invariant under the time-reparametrizations in (4.3). Hence, it requires imposing no *a priori* labelling of the foliation and is manifestly covariant and geometrical.

In fact, to elaborate on the final point above, let us introduce an everywhere well-behaved one-form field u_a proportional to the gradient of T

$$u_a = -N\nabla_a T \qquad \Leftrightarrow \qquad u_{[a}\nabla_b u_{c]} = 0\,. \tag{4.4}$$

The one form u_a is thus orthogonal to the leaves of the foliation Σ by Frobenius' theorem. If we furthermore require u_a to be unit normal everywhere in the spacetime, i.e.

$$u^2 \equiv u_a u_b \mathrm{g}^{ab} = -1\,, \tag{4.5}$$

then the above two relations are sufficient to determine the normalisation function N as

$$N = [-\mathrm{g}^{ab}(\nabla_a T)(\nabla_b T)]^{-1/2}\,. \tag{4.6}$$

In particular, the sign of N is fixed by choosing a parameter T that increases monotonically towards the future, which in turn ensures that u_a is *future directed*. To elaborate, a manifold with a globally ordered foliation, whose leaves are everywhere spacelike, is naturally *time orientable* by the converse of Lemma 8.1.1 of Wald (1984). This is because one may always construct a *continuous vector field* $u^a = g^{ab} u_b$ everywhere, whose existence is guaranteed by the global ordered and well-behaved nature of the foliation, without any need to refer to any particular time-parametrization. Thus spacetimes with an ordered foliation naturally admit a well-defined sense of past and future directedness, which is also preferred in our case.

From (4.6), the function N transforms under the T-reparametrizations of (4.3) as

$$N \mapsto \tilde{N} = (\mathrm{d}\tilde{T}/\mathrm{d}T)^{-1} N \,, \tag{4.7}$$

rendering u_a invariant under the transformation in (4.3). Therefore, quantities expressed in terms of u_a as well as other geometrical objects — e.g. tangents to curves in $\mathscr{M}$ — which can be defined without making any reference to any time-parametrization of the foliation will be automatically time-reparametrization invariant. The field u_a is referred to as the æther in the existing literature of Einstein-Æther theory (see Jacobson and Mattingly, 2001), and we will adopt this nomenclature below.

Having introduced the basic concepts related to manifolds that sport a preferred foliation, we can now turn our attention to a reparametrization invariant formulation of causality in terms of the æther.

4.2 CAUSALITY IN A FOLIATED MANIFOLD

In this Section, we wish to develop a framework that establishes causal relationships between events in spacetime without explicit reference to some specific time-parametrization of the foliation. For the reasons listed previously, we wish to stay close to the spirit of general relativity and use curves to determine causal relationships between events. As opposed to standard general relativity however, timelike and null curves do not exhaust the possibility of causal communication in our case, and hence we need to generalize the definition of *causal curves*. On a related and equally important note, the causal relationship between two events as specified by the metric is not sufficient for our purpose; rather such relationships will have to involve the ordered foliation structure in an essential way.

With these in mind, imagine a curve intersecting a stack of the ordered leaves of the foliation. If the curve does not 'turn around' and intersect the same leaf of the foliation more than once, the stack will naturally slice the curve in the same order as the leaves in the stack, imbibing the curve with the same ordering information carried by the foliation. In turn, to ensure that a curve 'does not turn back', it is sufficient to require that the tangent vector $\mathbf{t}^a$ of the curve maintains an inner-product with the æther one-form u_a which does not change sign. The following definition of causal curves is a formalization of the above idea.

Definition 1 (Causal and acausal curves)**.** A continuous and piecewise differentiable curve with tangent vector $\mathbf{t}^a$ will be called

$$\begin{aligned} \text{causal and future directed} \quad &\text{if} \quad (u \cdot \mathbf{t}) < 0 \quad \text{everywhere along the curve,} \\ \text{causal and past directed} \quad &\text{if} \quad (u \cdot \mathbf{t}) > 0 \quad \text{everywhere along the curve,} \\ \text{acausal} \quad &\text{if} \quad (u \cdot \mathbf{t}) = 0 \quad \text{everywhere along the curve.} \end{aligned}$$

Henceforth we will always work with curves that are continuous and piecewise differentiable. According to the above definition, a curve that is not causal is not necessarily acausal and vice versa. Curves that are piecewise causal and piecewise acausal certainly exist, but they are of little use in discussing causality. It is also worth pointing out that since the æther is more naturally defined as a one-form [see (4.4)], the metric associated with $\mathscr{M}$ is not actually necessary in order to determine the causal nature of a curve.

It will be useful to cite some examples of causal curves to develop some feeling for them. For instance, a curve generated by the integral curves of a vector which is locally proportional to the æther vector u^a, up to a function that does not change sign along the curve, is always causal by definition. Such curves are perhaps the most natural type of causal curves and will be called *preferred causal curves*. Next, a timelike geodesic of the metric

g_{ab} provides an example of a causal curve which is not preferred since its tangent vector is not aligned with the æther in general. On the other hand, a curve that is spacelike with respect to the metric g_{ab} (and hence not causal in general relativity) may still furnish an example of a curve that is causal in the present context. Using as an excuse that of constructing examples of such curves, let us discuss the notion of a *speed-c metric*. Such metrics were first introduced in Foster (2005) and will prove to be quite useful in the following. A speed-c metric $g^{(c)}_{ab}$ is a symmetric bilinear rank-two tensor built out of u_a, g_{ab} and a finite positive real number c as follows

$$\begin{aligned} g^{(c)}_{ab} &= g_{ab} - (c^2-1)u_a u_b \\ g^{ab}_{(c)} &= g^{ab} - (c^{-2}-1)u^a u^b \end{aligned} \qquad 0 < c < \infty\,. \tag{4.8}$$

One may verify that $g^{(c)}_{ab}$ is everywhere non-degenerate, and the corresponding inverse speed-c metric $g^{ab}_{(c)}$ can be given in terms u^a, g^{ab} and c as in (4.8). The speed-c metric gets its name from the fact that a point particle moving along a null curve of $g^{(c)}_{ab}$ has a local speed c as measured by an observer co-moving with the æther; in this sense g_{ab} is the 'speed-1 metric'. For $c > 1$, the speed-c metric has a propagation-cone that is strictly wider than that of g_{ab}. Therefore, there are timelike curves of $g^{(c)}_{ab}$ which are spacelike curves of g_{ab}. On the other hand, such curves are causal curves according to Definition 1.

Along with the notion of a causal curve discussed above, we also need to generalize the notions of the causal past and future of an event. Unlike in general relativity, we do not need to separate the notions of chronological and causal past/futures, since timelike and null curves do not play any significantly different roles in our discussion.

We will say that an event q is in the future (past) of another event p, if there exists a future (past) directed causal curve from p to q. The *causal future* of an event p, to be denoted by $J^+(p)$, is defined as the set of all events that can be reached from the event p by means of a future directed causal curve. We may analogously define $J^-(p)$, the *causal past* of an event

p, by replacing the 'future directed' with 'past directed' in the definition of $J^+(p)$.[2] We will require $p \notin J^\pm(p)$ as part of the definitions of $J^\pm(p)$ in order to avoid unphysical statements like 'p is connected to itself by a causal curve' etc. As simple extensions of the definitions of the causal past and future of a single event, one may define the causal future and past of a set of events $\mathscr{Q}$ as

$$J^+(\mathscr{Q}) \equiv \bigcup_{p \in \mathscr{Q}} J^+(p) \; , \qquad J^-(\mathscr{Q}) \equiv \bigcup_{p \in \mathscr{Q}} J^-(p) \; , \tag{4.9}$$

respectively. Finally, an event q will be simultaneous with a distinct event p if there exist no causal curve from p to q, i.e. if $q \notin J^\pm(p)$. Consequently, we have yet another representation of the leaf Σ_p [compare with (4.2)]

$$\Sigma_p = \{q \in \mathscr{M} \mid q \notin J^\pm(p)\} \; . \tag{4.10}$$

By the assumed connectedness (hence path-connectedness) of every leaf we can always connect any two events on a given leaf by an acausal curve, although not every curve that joins two distinct events on a given leaf is acausal.

In the following, sometimes it will be important to deal with just a subset of a leaf whose events are simultaneous. We will call such a set of events a *simset*, as a contraction of 'sim[ultanous] set'. More formally, a simset $\mathscr{S}_p$ of the event p is any open subset of Σ_p that contains the event p, i.e.

$$\mathscr{S}_p \subseteq \Sigma_p \; , \qquad p \in \mathscr{S}_p \; . \tag{4.11}$$

In particular Σ_p itself is a simset. A simset will be called a *proper simset* if it is a proper subset of some leaf. As it will become apparent in the following, the concept of simset is motivated by the concept of an achronal set of general relativity. However, while these concepts share some common features and mathematical properties, there are also some crucial differences

2 In principle, curves that are built by pieces that are causal future (or past) directed and others that are acausal are also allowed, as long as the causal portions are all either future or past directed. We didn't specify this possibility above to avoid confusion.

in their behaviour stemming from the different causal structures associated with them; we will emphasize these differences in the appropriate context below.

With concrete definitions of past, future and simultaneity laid down as above, we may now recast the requirement that the foliation be ordered into the *equivalent* statement that *the sets of past, simultaneous and future events of every event are mutually disjoint, i.e.*

$$\left.\begin{aligned} &J^-(p) \cap \Sigma_p = \emptyset \\ &J^+(p) \cap \Sigma_p = \emptyset \\ &J^-(p) \cap J^+(p) = \emptyset \end{aligned}\right. \qquad \forall p \in \mathscr{M} \ . \tag{4.12}$$

Conversely, a non-empty intersection among any two of the three sets $J^{\pm}(p)$ and Σ_p will necessarily imply the existence of a pair of events with more than one inequivalent causal relationship between them. One may also verify the transitive properties for causal relationships, thereby confirming consistency with a totally ordered foliation. As an immediate application of the above, one also has

$$\left.\begin{aligned} &J^+(q) \subset J^+(p) \\ &J^-(p) \subset J^-(q) \\ &J^+(q) \cap J^-(p) = \emptyset \end{aligned}\right. \qquad \forall q \in J^+(p) \ .$$

As already mentioned before, an ordered foliation is expected to imply a natural causality condition. Our setup is seemingly suitable for applying Theorem 8.2.2 in Wald (1984) at first sight, which proves that a spacetime is *stably causal* (i.e. possesses no closed timelike curves) if and only if it also globally admits a differentiable function with a past directed timelike gradient. However, on further reflection it becomes apparent that *global* existence of a differentiable function with a timelike gradient is not necessarily guaranteed by our setup. Furthermore, unlike general relativity, we also need to ensure that closed causal curves beyond timelike and null are ruled out. To that end, we provide the following result:

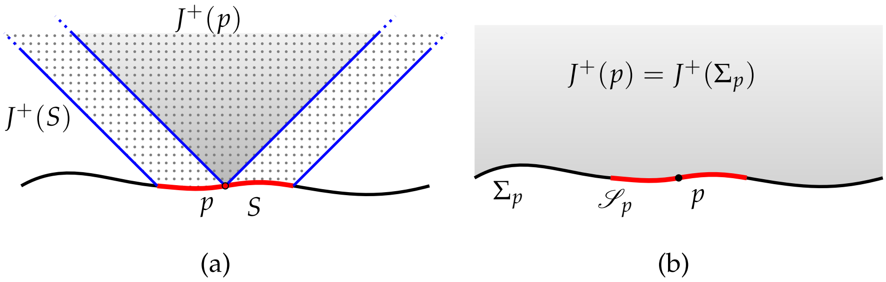


Figure 4: Difference between the notions of causal future in locally Lorentz invariant theories (A) and theories with a preferred foliation (B).

Proposition 1 (Causality condition). *No strictly future directed or strictly past directed causal curve may intersect a given leaf more than once.*

Proof. Suppose, a closed curve exists in $\mathscr{M}$, which is future-directed and causal between the leaves Σ_p and Σ_q, intersecting Σ_p at p_1 and Σ_q at q_1 such that $q_1 \in J^+(p_1)$. Since the curve is closed by assumption, it must intersect both Σ_p and Σ_q at least once more each, say, at events $p_2 \in \Sigma_p$ and $q_2 \in \Sigma_q$ respectively. Obviously, p_1 and p_2 are simultaneous since they reside on the same leaf, and so are q_1 and q_2 as well. But if the segment on the curve from q_2 to p_2 is not past directed anywhere, one has $q_2 \notin J^+(p_2)$. This contradicts the causal relationships between the pairs of events (p_1, q_1), (p_1, p_2) and (q_1, q_2), thereby violating condition (4.12). This proves that the curve must have a past directed causal segment.

Now, if a causal curve were to intersect a leaf more than once, one could join pairs of these events of intersection by acausal curves, in an obvious manner, to form closed curves with only future or past directed segments, but not both. Therefore by our previous result, every causal curve may intersect a given leaf at most once. □

Thus far, we have verified that our proposed definitions of past, future and simultaneity meet the most basic requirements of consistency. The rest of this work is devoted towards uncovering those unique features of causality in a foliated manifold which drastically contrast those of general relativity. As we already saw above, curves that are arbitrarily spacelike with respect to g_{ab} may still represent causal curves here. One of the rather remarkable consequences of the existence of such curves, our definition of future (past), and the causality condition as expressed in (4.12) is that the future (past) of every event is identical with the future (past) of the leaf on which the event resides or that of any simset of the leaf, i.e.

$$\begin{aligned} J^+(p) &= J^+(\Sigma_p) = J^+(\mathscr{S}_q) \\ J^-(p) &= J^-(\Sigma_p) = J^-(\mathscr{S}_q) \end{aligned} \qquad \forall p \in \mathscr{M}, \quad \forall q \in \Sigma_p \, . \tag{4.13}$$

We can now make some comments and observations on the open/closedness of the sets $J^{\pm}(\Sigma_p)$ and related properties of their respective closures. Consider the set $J^+(\Sigma_p)$ to begin with. Since the whole spacetime is open by assumption, $J^+(\Sigma_p)$ cannot contain any 'boundary events', i.e. every event $q \in J^+(\Sigma_p)$ should admit at least one open neighbourhood $\mathscr{O}_q \subseteq J^+(\Sigma_p)$; more formally, one may invoke the results of Theorem 8.1.2 of Wald (1984) (see also Proposition 2.8 of Penrose (1972) or Lemma 14.2 of O'Neill (1983)) in order to construct a proof of this. Therefore $J^+(\Sigma_p)$ is an open set.

The fact that $J^+(\Sigma_p)$ is open can also be deduced in a more intuitive fashion as follows: the speed-c metric $g^{(c)}_{ab}$ of (4.8) allows us to *formally* associate an open set $I^+_{(c)}(p)$ – the general relativistic chronological future of p constructed with $g^{(c)}_{ab}$ – at every event $p \in \mathscr{M}$. The collection $\{I^+_{(c)}(p) \mid c > 0\}$ then forms an open cover of $J^+(p)$ such that $J^+(p) = \bigcup_{c>0} I^+_{(c)}(p)$. Therefore $J^+(p)$, and hence $J^+(\Sigma_p)$ by virtue of (4.13), are open. We should emphasize that the open sets $I^+_{(c)}(p)$ have been used as pure mathematical objects in the above argument; in particular, they have no physical significance in regards to the causality of the backgrounds. However, the proof does rest on the intuitive picture that in a locally Lorentz invariance violating geometry, causal curves are no longer contained in any fixed propagation cones, and that the leaves of the foliation are the result of 'opening up' of the local propagation cones to their maximum in their attempt to contain these causal curves within them.

One may likewise argue that $J^-(\Sigma_p)$ is an open set. Furthermore, from the openness of $J^{\pm}(\Sigma_p)$ and the spacetime being a Hausdorff manifold, it is straightforward to deduce that for every pair of distinct non-simultaneous events $p, q \in \mathscr{M}$ such that $q \in J^+(p)$, there must exist disjoint open neighbourhoods $\mathscr{O}_p$ of p and $\mathscr{O}_q$ of q such that every event in $\mathscr{O}_q$ is in the future of every event in $\mathscr{O}_p$.

Given the unique causal relationship between every pair of events $p, q \in \mathscr{M}$, we then have the following three mutually exclusive possibilities: q must be either in the past of p, or be simultaneous with p, or else be in the future of p. One may summarize this as

$$\mathscr{M} = J^+(\Sigma_p) \cup \Sigma_p \cup J^-(\Sigma_p)\,, \qquad \forall p \in \mathscr{M}\,. \tag{4.14}$$

As a trivial consequence of the above relation, every leaf is a closed set in $\mathscr{M}$. This is however expected, as every leaf is essentially composed of 'boundary points'. In other words, for every event $q \in \Sigma_p$, every open neighbourhood $\mathscr{O}_q$ of q contains events which are on the leaf as well as events which are not on the leaf.

From the above discussions, it also follows immediately that the spacetime $\mathscr{M}$ cannot be compact in every direction without violating our causality condition (4.12). For suppose $\mathscr{M}$ were compact, i.e. every open cover of $\mathscr{M}$ had a finite subcover. Consider now the open cover given by $\{J^+(\Sigma_p)|\, \forall p \in \mathscr{M}\}$. By the assumed compactness of $\mathscr{M}$, this open cover should have a finite subcover, say $\{J^+(\Sigma_{p_1}), \cdots, J^+(\Sigma_{p_n})\}$, for some finite integer n, such that $\mathscr{M} = \bigcup_{i=1}^n J^+(\Sigma_{p_i})$. Furthermore, we may assume without any loss of generality that the events $\{p_1, \cdots, p_n\}$ are ordered in a chronological fashion so that p_1 is not in the future of any of the other events. But then, $J^+(\Sigma_{p_i}) \subseteq J^+(\Sigma_{p_1})$ for all $i \neq 1$, which would imply $\mathscr{M} = J^+(\Sigma_{p_1})$, and hence $p_1 \notin \mathscr{M}$. This is a contradiction; therefore, $\mathscr{M}$ cannot be compact. One may note that our argument is a direct adaptation of similar arguments in general relativity (see, e.g. Proposition 6.4.2 of Hawking and Ellis (1973) or Lemma 10 of O'Neill (1983)).

Finally, we may close both $J^{\pm}(\Sigma_p)$ in $\mathscr{M}$ simply by appending the leaf Σ_p to the respective sets

$$\begin{aligned} \overline{J^{+}(\Sigma_p)} &= J^{+}(\Sigma_p) \cup \Sigma_p = J^{-}(p)^c \quad \forall p \in \mathscr{M} \ , \\ \overline{J^{-}(\Sigma_p)} &= J^{-}(\Sigma_p) \cup \Sigma_p = J^{+}(p)^c \quad \forall p \in \mathscr{M} \ , \end{aligned} \tag{4.15}$$

where $\mathscr{Q}^c \equiv \mathscr{M} \setminus \mathscr{Q}$ is the complement of $\mathscr{Q}$ in $\mathscr{M}$. In particular, the relationship between the closure of the future (past) and the complement of the past (future) follows directly from (4.14). Given the closures, the boundaries $\partial J^{\pm}(\Sigma_p)$ of the past and the future sets of Σ_p in $\mathscr{M}$ are then given by

$$\partial J^{+}(\Sigma_p) = \partial J^{-}(\Sigma_p) = \Sigma_p \ , \qquad \forall p \in \mathscr{M} \ . \tag{4.16}$$

4.3 ASYMPTOTICS

Our final goal is that of uncovering an appropriate definition of black hole and of the corresponding spacetime configuration. As is the case in general relativity, the notion of an asymptotic region will be central to the discussion; only in the presence of such a notion one may precisely make sense of 'moving far away' from the black hole and be able to claim that 'nothing can escape to infinity' from the region of spacetime beyond an event horizon. Hence, before proceeding any further, we will devote this Section to a discussion about how to properly treat asymptotics in our context. Our focus will be on the simplest case of asymptotics, which is a suitable generalization of the concept of asymptotic flatness.

In spherically symmetric asymptotically flat geometries (see e.g. Barausse et al., 2011) the notion of infinity comes very naturally in terms of the areal radial coordinate. When considering configurations which enjoy less symmetries on the other hand, one cannot follow a similar prescription. In what follows our primary goal will be to formalise the notion of an *asymptotic region* of a foliated spacetime — or of part of one — beyond any particular symmetries, along with the associated notion of a boundary at infinity.

In general relativity, as an outgrowth of the seminal work of Bondi et al. (1962); Sachs (1962); Penrose (1963, 1965), we have a precise notion of what it means for a spacetime to be *asymptotically flat at null infinity*. The question motivating the studies in this direction was that of understanding what defines an isolated gravitating system; the notion of infinity thus formulated allows one to place an observer at infinity, abstractly yet consistently, with respect to whom the said gravitating system appears completely isolated.

A different line of investigation started with the attempt to formulate an initial value problem for general relativity (see Arnowitt et al., 1962, 2008). One of the notable results of this line of investigation was uncovered by Geroch, who formalized the notion of *asymptotic flatness at spatial infinity* (Geroch, 1972) in terms of the asymptotic behaviour of initial data on a Cauchy surface. Eventually, such seemingly different formulations of infinity in asymptotically flat spacetimes were unified, in particular starting with the work of Ashtekar and Hansen (1978) (see also Ashtekar, 1980). Similar studies leading to suitable definitions of asymptotic structures of other kinds of spacetimes (e.g. anti-de Sitter; see Ashtekar and Magnon, 1984; Henneaux and Teitelboim, 1985) have been performed.

One obvious yet important upshot of these studies in the context of general relativity is that for every type of direction along which one may wish to travel in spacetime, there is a corresponding notion of infinity, provided that it is possible to reach infinity along this direction. For instance, in the case of asymptotically flat spacetimes one has the notions of (i) future and past timelike infinities, denoted by $i^{\pm}$ respectively, where one may end up by travelling along future and past directed timelike curves respectively, (ii) future and past null infinities, denoted by $\mathscr{I}^{\pm}$ respectively, where one may end up by travelling along future and past directed null curves respectively, and finally (iii) spacelike infinity, denoted by i^0, where all spacelike curves finish. Such overall structure is expected in the context of general relativity due to the significance of timelike and null curves in determining the causal structure of the spacetime.

In the foliated spacetimes we are studying in this work we only have a single variety of causal curves. Consequently, the only notion of infinity that we need to formalise is the one associated with such causal curves. In principle, of course, one may still talk about some asymptotic structure similar to general relativity with respect to a speed-c metric of (4.8) for some fixed value of c. However, such structures can only be relevant for perturbations that are restricted to propagate inside or on the null cones of the chosen speed-c metric.

In our setting, on the other hand, we have a fundamental departure from such relativistic asymptotic structures, since signals can propagate arbitrarily fast and cannot be contained permanently within the null cone of any speed-c metric irrespective of how large c is. Intuitively, such arbitrarily fast propagations are expected to end up at some appropriately defined spatial infinity. As we will try to argue below, there is a simple generalization of the approach discussed in Geroch (1972) that leads to a proper definition of infinity suitable for our needs.

In general relativity the asymptotic structure of spacetimes is studied by *conformally compactifying* the physical spacetime $\mathscr{M}$ into a larger compact manifold with boundaries $\tilde{\mathscr{M}}$, where the physical metric g_{ab} is related to the metric $\tilde{\mathsf{g}}_{ab}$ on $\tilde{\mathscr{M}}$ through a conformal transformation $\tilde{\mathsf{g}}_{ab} = \Omega^2 \mathsf{g}_{ab}$ with $\Omega > 0$. It is also easy to show that such a transformation will naturally induce a corresponding conformal transformation on the three-metric induced by g_{ab} on an initial data set, allowing for a conformal compactification of the latter. This serves as a starting point of Geroch's formulation of spatial infinity (Geroch, 1972).

In order to mimic the above procedure, let us begin with defining the three dimensional metric (projector) p_{ab} and its inverse p^{ab}, induced by the full spacetime metric g_{ab} on each leaf of the foliation as

$$\mathsf{p}_{ab} = \mathsf{g}_{ab} + u_a u_b \,, \qquad \mathsf{p}^{ab} = \mathsf{g}^{ab} + u^a u^b \,. \tag{4.17}$$

For every leaf Σ_p with the induced metric p_{ab} as defined above (in the relevant part of the spacetime), we wish to obtain a connected, Hausdorff and compact three-manifold $\tilde{\Sigma}_p$ into which Σ_p is to be embedded, such that the three-metric $\tilde{\mathsf{p}}_{ab}$ on $\tilde{\Sigma}_p$ is related to p_{ab} via a conformal transformation

$$\begin{aligned} \mathsf{p}_{ab} &\mapsto \tilde{\mathsf{p}}_{ab} = \Omega^2 \mathsf{p}_{ab} \\ \mathsf{p}^{ab} &\mapsto \tilde{\mathsf{p}}^{ab} = \Omega^{-2} \mathsf{p}^{ab} \end{aligned} \qquad \Omega > 0\,, \tag{4.18}$$

for some function Ω defined on the spacetime with some appropriate asymptotic behaviour. Note, as an aside, that the unit maps — or projectors on the hypersurface — ${\mathsf{p}^a}_b$ and ${\mathsf{p}_a}^b$ remain unaffected by the above transformations. Topologically the above procedure is equivalent to a *one-point compactification* of the leaf; namely, we append to the leaf Σ_p an event $\mathfrak{i}_p$ such that the 'larger' manifold

$$\tilde{\Sigma}_p = \Sigma_p \cup \{\mathfrak{i}_p\} \tag{4.19}$$

is compact. We will then call $\tilde{\Sigma}_p$ a *conformal extension* of the leaf Σ_p. A standard result from topology states that every locally-compact non-compact Hausdorff space has a unique one-point compactification (Munkres, 2000). Every leaf of the foliation can in principle be compactified using this procedure, as it is a Hausdorff manifold by assumption. More importantly, the procedure of conformal compactification not only ensures a unique topological structure on $\tilde{\Sigma}_p$, but in fact a unique differential structure on it (see Geroch, 1970, 1972). In accordance with standard practice, the event $\mathfrak{i}_p$ will be called *the point at (spatial) infinity on the leaf* Σ_p.

In general relativity, the next round of business usually involves postulating appropriate behaviour of the metric and the function Ω at the point at infinity such to guarantee a suitable asymptotic behaviour of the spacetime at spatial infinity. We will henceforth consider the simplest asymptotic behaviour, namely *asymptotic flatness*. Intuitively, an asymptotically flat spacetime is characterized by sufficiently fast fall-off of all matter fields such that asymptotically the spacetime appears empty and Minkowskian. However, in the present context, we also need to worry about the behaviour of the foliation, or equivalently the æther, at infinity. In fact as an integral part of our background, being responsible for defining the foliation structure, the æther cannot be treated as some matter field anymore.

In the discussion that follows, global Minkowski spacetime with a constant æther aligned with a timelike Killing vector will play a very similar role to that played by global Minkowski spacetime in general relativity. Anticipating its importance we will henceforth denote such a spacetime as a *trivially foliated flat spacetime*.[3] Such spacetime is maximally symmetric, with both the metric and the æther satisfying all of the available Killing symmetries. In particular, one may always choose standard Minkowski coordinates in which the æther is given by $u_a = -\nabla_a t$, where t is the Minkowski time coordinate.

In more general spacetimes, a leaf Σ_p is said to admit a *trivially foliated asymptotically flat end* if one may conformally extend the leaf by appending a point at infinity $\mathfrak{i}_p$ to it [recall (4.19)] such that the asymptotic behaviour of the spacetime and æther approaches that of a trivially foliated flat spacetime as one approaches $\mathfrak{i}_p$. Formally, this implies that two separate conditions need to be satisfied. The three-metric p_{ab} should satisfy the usual general relativistic conditions of asymptotic flatness at spatial infinity, thoroughly discussed in Geroch (1972). Additionally, the æther should also have the appropriate asymptotic behaviour, which means that it should align with an asymptotic timelike Killing vector at infinity.

In order to satisfy this last requirement, one can then introduce a local rescaling of the æther as follows

$$\begin{aligned} u_a &\mapsto \tilde{u}_a = \Omega_u u_a \\ u^a &\mapsto \tilde{u}^a = \Omega_u^{-1} u^a \end{aligned} \qquad \Omega_u > 0\,, \tag{4.20}$$

where $\Omega_u \neq \Omega$ is some function on the spacetime. The above transformation of the æther naturally preserves the foliation structure, i.e. $\tilde{u}_a$ is hypersur-

3 It is straightforward to show that Hořava gravity admits such backgrounds as vacuum solutions.

face orthogonal with respect to the same foliation structure as u_a,[4] and the unit norm constraint (4.5) is also maintained. Furthermore, since $\Omega_u \neq \Omega$ in general, (4.18) and (4.20) leads to a *local disformal transformation* of the metric given by

$$\begin{aligned} \mathrm{g}_{ab} &\mapsto \tilde{\mathrm{g}}_{ab} = \Omega^2 \mathrm{g}_{ab} + (\Omega^2 - \Omega_u^2) u_a u_b = \Omega^2(\mathrm{g}_{ab} - [c(x)^2 - 1] u_a u_b) \, , \\ \mathrm{g}^{ab} &\mapsto \tilde{\mathrm{g}}^{ab} = \Omega^{-2} \mathrm{g}^{ab} + (\Omega^{-2} - \Omega_u^{-2}) u^a u^b = \Omega^{-2}(\mathrm{g}^{ab} - [c(x)^{-2} - 1] u^a u^b) \, , \end{aligned} \tag{4.21}$$

with $c(x) \equiv \Omega_u \Omega^{-1}$; as before, the unit maps ${\delta^a}_b$ and ${\delta_a}^b$ are unaffected by the transformations. The standard conformal transformation of the metric is recovered when $\Omega_u = \Omega$, or equivalently, when $c(x) = 1$. One may also view the disformal transformations of (4.21) as a local generalization of the global field redefinitions introduced in Foster (2005). From this perspective, a disformal transformation locally maps the four-metric conformally to some speed-c metric (4.8) at the same location, i.e. one chooses $c = c(x_0)$ at the location $x = x_0$.

By specifying a suitable asymptotic behaviour of Ω and Ω_u at $\mathfrak{i}_p$, one may appropriately generalize the conditions postulated in Geroch (1972) and formally define the notion of a trivially foliated asymptotically flat end of a leaf. However, it is important to note that the aforementioned asymptotic behaviour of Ω and Ω_u will be sensitive to the specific theory under consideration. For this reason, and given the scope of this thesis, we will not venture into the technical details of these conditions.

According to the above prescription, every leaf in a trivially foliated flat spacetime admits a trivially foliated flat end, as consistency demands. Moving on to more general spacetimes, one may formally attach a suitable notion of asymptotic region to (a part of) a foliated spacetime $\mathscr{M}$ as follows: an open region $\langle\langle \mathscr{M} \rangle\rangle \subseteq \mathscr{M}$ will be said to admit a *trivially foliated asymptotically flat end*, if every leaf $\Sigma_p \subset \langle\langle \mathscr{M} \rangle\rangle$ has a trivially foliated asymptotically flat

4 This is obvious from the fact that $\tilde{u}_a$, as defined above, is proportional to a one-form which is orthogonal to the same set of hypersurfaces.

end in the sense defined above. In order to avoid any possible issues in our future constructions, we will assume henceforth that $\langle\langle \mathscr{M} \rangle\rangle$ is the maximal open region to admit a trivially foliated asymptotically flat end. The region $\langle\langle \mathscr{M} \rangle\rangle$ of a trivially foliated flat spacetime is identical with $\mathscr{M}$ itself, but this is not true for more general spacetimes. Indeed, the fact that a spacetime may have regions beyond $\langle\langle \mathscr{M} \rangle\rangle$ is at the heart of the concept of a black hole, as will be seen in Section 5.2.1 of Chapter 5. Finally, by 'stringing together' all the points at infinity, we can formally define the *asymptotically flat end* $\mathscr{I}$ of $\langle\langle \mathscr{M} \rangle\rangle$ as

$$\mathscr{I} = \bigcup_{p \in \langle\langle \mathscr{M} \rangle\rangle} \mathrm{i}_p \,. \tag{4.22}$$

The above equation thus defines the sought after notion of an asymptotically flat boundary at infinity.

The above definition essentially ensures that the region $\langle\langle \mathscr{M} \rangle\rangle$ of a general spacetime $\mathscr{M}$ admits an asymptotic region along with a trivially foliated flat end if a suitably chosen open neighbourhood of $\mathscr{I}$ in $\langle\langle \mathscr{M} \rangle\rangle$ 'resembles' a similarly chosen open neighbourhood of $\mathscr{I}$ of a trivially foliated flat spacetime; the latter can be easily identified through straightforward extensions of standard conformal techniques of general relativity. Delving deeper into such matters is, however, beyond the scope of the present work.

4.4 CAUSAL DEVELOPMENT AND THE CAUCHY HORIZON

The notions of causal past and future introduced above allow us to determine whether or not a given region of spacetime can causally affect or influence another region. More precisely, we may define the *future domain of influence* of a set of events $\mathscr{Q}$ as the set of all events in $\mathscr{M}$ which can be causally influenced by events in $\mathscr{Q}$. Since causal influence can only propagate along causal curves, the future domain of influence of $\mathscr{Q}$ is identical with its causal future. We now turn to a related but slightly more involved question, namely, given a set of events $\mathscr{Q}$ how much of the spacetime —

and of the evolution of the fields living on it — can we predict based on the information associated with $\mathscr{Q}$? The *future domain of dependence* of $\mathscr{Q}$ is a subset of the corresponding domain of influence, consisting of events which are causally influenced *only* by events in $\mathscr{Q}$. As such, the future domain of dependence of $\mathscr{Q}$ consists of events which are *predicted with and only with* the information associated with $\mathscr{Q}$. There are of course analogous concepts when the words 'future' and 'prediction' above are replaced with 'past' and 'retrodiction', respectively. Needless to say, all such notions are very natural adaptations of the corresponding ones in general relativity.

The actual process of a prediction requires dynamical equations that can evolve initial information. The details of such prediction through solving equations of motion lie beyond our current goals. In fact, as we have already mentioned, we are willing to assume well-posedness of the initial value problem. However, there is one characteristic of the equations that is crucial for the appropriate definition of the domain of dependence which requires special attention; this is whether the corresponding problem consists only of elliptic constraint equations and hyperbolic evolution equations or whether it also involves additional elliptic equations that do not constitute constraints.

Recall our assumption has been from the onset that there is a preferred foliation that determines the causal structure of the spacetime. Physically, this means that events that lie on the same slice of the foliation are simultaneous and share the same future (4.13). Mathematically, this means that by assumption the initial value problem is well-posed only in this foliation. We are faced then with at least two distinct options:

(a) certain constraints relate initial data for the dynamical variables on a given slice of the foliation and the hyperbolic equations determine the evolution of the dynamical variables;

(b) in addition to the above, the theory contains variables that are not determined by the dynamical equation, but they should instead be determined on every slice by means of solving an elliptic equation *on* the slice itself.

In the second case, one will need asymptotic or boundary conditions — potentially periodic in certain topologies — in order to have a well-formulated problem.

For example, the infrared limit of projectable Hořava gravity (Hořava, 2009b; Sotiriou et al., 2009a,b) falls under case (a). On the other hand, as has been mentioned in the introduction and discussed in Blas and Sibiryakov (2011), the most general non-projectable Hořava theory falls under case (b), (see Bhattacharyya et al. (2016a) for a detailed discussion). Here and for what concerns the notion of development, we choose to consider only case (b). Our main motivation for doing so is the following: it has been conjectured in Blas and Sibiryakov (2011), based on intuition from perturbative decoupling calculations around spherically symmetric solutions, that universal horizons are Cauchy horizons in non-projectable Hořava gravity. We will be able, once we have a formal definition of universal horizon, to check whether we can infer more information about this conjecture. To do so on the other hand we will need a definition of Cauchy horizon.

In the previous Section we have defined a suitable notion of boundary at infinity, so the next step shall be to define an appropriate notion of development that will depend on both initial and boundary data. However, a boundary at infinity is clearly not the only boundary one can have. So, before going any further we will discuss other types of boundaries that may be relevant here. To motivate the problem, we could perhaps start with an example of an 'artificial boundary': given a general foliated spacetime one may wish to consider the evolution of some fields in some *restricted region* of the spacetime. For example, such a restricted region may be a cube or a shell of a fixed size in a trivially foliated flat spacetime, and the boundary of the region may impose boundary conditions on the fields that live inside.[5]

5 Possibilities traditionally considered in GR settings include reflecting or Dirichlet boundary conditions.

Now, suppose we have an appropriate set of coupled hyperbolic and elliptic partial differential equations of motion for some fields, as mentioned at the beginning of this section. Let $\mathscr{S}_p \subset \Sigma_p$ be a proper simset of some leaf Σ_p, with a boundary $\partial\mathscr{S}_p$ such that $\overline{\mathscr{S}_p} \equiv \mathscr{S}_p \cup \partial\mathscr{S}_p$ is the closure of the simset in the leaf, and suppose that we are provided with some appropriate initial conditions for these fields on $\mathscr{S}_p$ as well as suitable boundary conditions on $\partial\mathscr{S}_p$. As already emphasized though, the question of future/past causal development of such initial and boundary data associated with $\mathscr{S}_p$ and $\partial\mathscr{S}_p$ respectively also requires a specification of some appropriate boundary conditions in the past/future of $\mathscr{S}_p$, and the boundary in question should exist as a suitable chronological extension of $\partial\mathscr{S}_p$. Consider, thus, a set of causal vectors $\boldsymbol{b}^a$ defined everywhere on $\partial\mathscr{S}_p$ satisfying $(u \cdot \boldsymbol{b}) < 0$. The integral curves of $\boldsymbol{b}^a$ will then define the causal boundary $\mathscr{B}$ as a 'tube' with base on $\mathscr{S}_p$ (extending on both sides of Σ_p). One should then be able to specify suitable initial conditions on $\mathscr{S}_p$ and boundary conditions on $\mathscr{B}$ as the first step of setting up a well-formulated initial-boundary value problem.

On the other hand, it is usually more interesting to consider the evolution of fields — including the metric — in the whole of the spacetime, starting from the data associated with some initial leaf Σ_p. Indeed, this is the appropriate scenario for studying the dynamical construction of the spacetime itself. As already emphasized several times, this is not possible in the present case just with the initial data associated with the initial leaf Σ_p. Rather, one needs to consider appropriate boundaries to be able to associate boundary data for the elliptic equations. Furthermore, when complete leaves act as initial data surfaces, such boundaries can only be suitable *conformal boundaries* of the spacetime, i.e. those which mark the true 'ends' of the spacetime. This fact makes the issue of causal development in the present scenario remarkably different from that of general relativity.

We already considered the issue of conformal extending the spacetime $\mathscr{M}$ in Section 4.3, where we introduced the notion of the conformal boundary at infinity $\mathscr{I}$. However, it may be the case that a consistent conformal extension of the full spacetime not only admits the boundary at infinity $\mathscr{I}$ but also additional conformal boundaries distinct from $\mathscr{I}$. In what follows, $\mathscr{B}_{\mathrm{C}}$ will denote the collection of all such possible conformal boundaries disjoint from $\mathscr{I}$ which mark the 'remaining ends' of the spacetime, such that

$$\partial\mathscr{M} = \mathscr{I} \cup \mathscr{B}_{\mathrm{C}} \tag{4.23}$$

denotes the complete boundary of the spacetime, and

$$\mathscr{I} \cap \mathscr{B}_{\mathrm{C}} = \emptyset\,. \tag{4.24}$$

Of course, $\mathscr{B}_{\mathrm{C}}$ will be empty if $\mathscr{I}$ is the only conformal boundary one may need to consider. We may stress that, by definition, $\mathscr{B}_{\mathrm{C}}$ — and hence also the full boundary $\partial\mathscr{M}$ — is not part of the spacetime, just like $\mathscr{I}$.

From the discussion above, it is apparent that we have a rather large set of possibilities when it comes to defining boundaries, and this ultimately depends on the problem at hand. Correspondingly, a broad definition of boundaries which can encompass all the interesting and physically consistent cases is necessary; only in this way we are able to propose a unified definition of the domain of dependence. For the sake of a coherent presentation, however, it seems prudent to postpone a formalization of the notion of a boundary until we are ready to formally define the domain of dependence as well. Until then, it will suffice to retain an intuitive notion of a boundary as *a part of the spacetime or its conformal extension (or a combination of both if and when appropriate) which manages to close every leaf that it encompasses*.

Apart from issues related to boundaries as addressed above, we also need to consider the evolution of initial data — associated with a set of events that lie on the same leaf — via hyperbolic equations. As already discussed, this is the only meaningful choice in our setting since only such events are simultaneous in a preferred sense and a set of such events is the sensible analogue of the notion of an achronal set of general relativity. Thus, given an arbitrary simset $\mathscr{S}_p \subseteq \Sigma_p$, one might be tempted to follow the lore of general relativity and define the future domain of dependence of $\mathscr{S}_p$ — at least in regards to the hyperbolic sector of the evolution — as the set of all events $q \in J^+(\Sigma_p)$ such that all past directed causal curves emanating from q intersects $\mathscr{S}_p$ when sufficiently extended. However, this naïve adaptation of the definition of general relativity is rather incomplete for a number of reasons.

First of all, it clearly disregards the existence and influence of any boundary condition, whose necessity has already been emphasized. Consider, for example, the case where there is a boundary at infinity. Signals from the boundary can actually influence any event of a slice that reaches the boundary, due to the existence of an elliptic mode. Secondly, the above proposal suffers from a more technical drawback. Consider an event $q \in J^+(\Sigma_p)$. There will always be past-directed causal curves that pass from q but fail to reach $\mathscr{S}_p$ simply because they cannot be extended to do so. This problem already exists in general relativity and is dealt with by introducing the notion of curve extendibility. To elaborate, given a past directed causal curve $\lambda(\tau)$, we may call an event q to be a *past endpoint* of $\lambda(\tau)$ if for every neighbourhood $\mathscr{O}_q$ of q, there exists a τ_0 such that $\lambda(\tau) \in \mathscr{O}_q$ for all $\tau > \tau_0$. Furthermore, a past directed causal curve with no past endpoint may be called past inextendible. In general relativity the definition of the future domain of dependence is appropriately phrased in terms of past inextendible curves and this solves the problem. However, here the problem is more acute as one can have curves that are inextendible in the sense defined above but still asymptote to a particular leaf without ever intersecting it.

The following examples of curves constructed in a trivially foliated flat spacetime should illustrate this point effectively. Consider Minkowski spacetime and let Minkowski time t label the leaves of the 'preferred' (but otherwise trivial) foliation. Then the following curves

$$\{t, \sin(a\pi/t)\}\,, \qquad \{t, e^{(a/t-1)} - 1\}\,, \qquad \{t, [e^{(a/t-1)} - 1]\sin(a\pi/t)\}\,, \tag{4.25}$$

are causal everywhere, yet asymptote to the leaf defined by $t = 0$. Here a is a constant which sets the unit, and we have suppressed the y and z coordinates for convenience; they can be set to zero for the purpose of this illustration. These examples hopefully clarify that, in the present scenario, certain causal curves cannot be arbitrarily extended because they get trapped near a leaf instead of near an event.

It is therefore clear that we need to go beyond the notion of an endpoint of a curve in order to provide a definition of curve inextendibility suitable to our purposes. To that end, we will introduce the notions of *past and future endleaves* as follows: a leaf Σ_p will be called a *past endleaf* of a past directed causal curve $\lambda(\tau)$ if

$$\lambda(\tau) \cap (\mathscr{O}_p \cap J^+(\Sigma_p)) \neq \emptyset \quad \text{and} \quad \lambda(\tau) \cap (\mathscr{O}_p \cap J^-(\Sigma_p)) = \emptyset\,,$$

for all $\tau > \tau_0$ and for every open neighbourhood $\mathscr{O}_p \supset \Sigma_p$. Similarly, a leaf Σ_p is called a *future endleaf* of a future directed causal curve $\lambda(\tau)$ if

$$\lambda(\tau) \cap (\mathscr{O}_p \cap J^-(\Sigma_p)) \neq \emptyset \quad \text{and} \quad \lambda(\tau) \cap (\mathscr{O}_p \cap J^+(\Sigma_p)) = \emptyset\,,$$

for all $\tau > \tau_0$ and for every open neighbourhood $\mathscr{O}_p \supset \Sigma_p$. Note that according to the formal definition presented above a causal curve with a conventional endpoint — in the sense of general relativity — also admits an endleaf; in this case, the endleaf is the leaf which contains the endpoint. However, the converse is not always true. It might also be worth stressing that curves with endleaves but without endpoints, such as those illustrated in the examples of (4.25), cannot be causal from the perspective of general relativity.

With this in mind, we can now appropriately generalize the notion of past/future inextendible curves from general relativity. A causal curve $\lambda(\tau) \in \mathscr{M}$ with $\tau \in [0, \infty)$ and $\lambda(0) \in \mathscr{M}$ will be called *(future) past inextendible* if it has no (future) past endleaves.

We are at this point in the position to propose a precise and consistent definition of the domain of dependence that is modelled after the corresponding definition in general relativity, but takes into account the very different causal structure of spacetimes with a preferred foliation as well as the importance of boundary conditions.

4.4.1 *Domains of dependence and Cauchy horizons*

In what follows, we will denote with $\tilde{\Sigma}_p$ the conformal extension containing *all* possible conformal boundary events of a given leaf Σ_p. Additionally, suppose we are given a simset $\mathscr{S}_p \subseteq \Sigma_p$, such that its boundary $\partial\mathscr{S}_p$ is either in Σ_p or in $\tilde{\Sigma}_p$. We will denote, by $\overline{\mathscr{S}_p} \equiv \mathscr{S}_p \cup \partial\mathscr{S}_p$, the closure of $\mathscr{S}_p$ in Σ_p or $\tilde{\Sigma}_p$, as appropriate. Furthermore, suppose we are given a subset $\mathscr{B}$ of the spacetime or its conformal extension such that $\partial\mathscr{S}_p \subset \mathscr{B}$. We have then the following result.

Definition 2 (Future and past domains of dependence)**.** An event $q \in J^+(\Sigma_p)$ is in the future domain of dependence $D^+(\mathscr{S}_p, \mathscr{B})$ of $\mathscr{S}_p$ and $\mathscr{B}$, if

(i) either a simset of the leaf Σ_q is closed by $\Sigma_q \cap \mathscr{B}$ (or by $\tilde{\Sigma}_q \cap \mathscr{B}$ if appropriate), or the leaf Σ_q itself if closed by $\tilde{\Sigma}_q \cap \mathscr{B}$, and the same condition holds true for every leaf in $J^-(\Sigma_q) \cap J^+(\Sigma_p)$, and

(ii) every past inextendible causal curve through q either intersects $\mathscr{S}_p$ or reaches $\mathscr{B}$.

Similarly, the past domain of dependence $D^-(\mathscr{S}_p, \mathscr{B})$ of $\mathscr{S}_p$ and $\mathscr{B}$ is defined to include all events $q \in J^-(\Sigma_p)$ such that condition (i) holds for Σ_q as well as all leaves in $J^+(\Sigma_q) \cap J^-(\Sigma_p)$, and the words 'past inextendible' in condition (ii) is replaced with 'future inextendible'.

We will regard the set $\mathscr{B}$ to form part of the *boundary* of the domains of dependence $D^\pm(\mathscr{S}_p, \mathscr{B})$.

It is worth emphasizing that Definition 2 not only defines the domain of dependence of a simset but also formalizes — as promised — the notion of boundary with respect to which boundary conditions are set, without which the concept of causal development would be vacuous in the present context. In particular, condition (i) ensures *continuity of the boundary*, i.e. the condition that a boundary cannot have any 'holes' and/or other kinds of discontinuities, as one would reasonably expect.[6] Indeed, for any such discontinuities, it is not clear how boundary conditions could be suitably prescribed for all time, and in turn how one may consistently talk about evolution. Condition (i) is thus intimately related to the presence of elliptic equations whose solutions depend crucially on boundary data. On the other hand, condition (ii) is modelled after the corresponding definition of general relativity, but also incorporates the possible influence from the boundary, and is essential for a consistent evolution of all kinds of modes, both elliptic and hyperbolic. In the context of future development, if any leaf $\Sigma_{q'} \subset J^{-}(\Sigma_q) \cap J^{+}(\Sigma_p)$ violated condition (i) — or otherwise said the boundary $\mathscr{B}$ had holes such that no simset of $\Sigma_{q'}$ could be closed by $\Sigma_{q'} \cap \mathscr{B}$ — one could construct a past inextendible causal curve from q through such a hole all the way to $\Sigma_p \setminus \overline{\mathscr{S}}_p$, thereby violating condition (ii). Note, in this regard, that condition (ii) has to holds for all events in the simset of $\Sigma_{q'} \subset J^{-}(\Sigma_q) \cap J^{+}(\Sigma_p)$ which is closed by $\Sigma_{q'} \cap \mathscr{B}$, since otherwise the same condition would fail to hold for the event q itself. In other words, as consistency demands, Definition 2 in all its entirety ensures that if an event is in the future development of some simset, then so should be the events in its past. Similar observations apply to past developments as well.

As an immediate consequence of the above definitions, when the development of an entire leaf Σ_p is under consideration, with $\partial\mathscr{M}$ [as defined in (4.23)] or parts of it serving as the appropriate boundary, all proper simsets of Σ_p have empty development since they violate all the criteria in Definition 2.

It should also be noted that Definition 2 allows for a fairly unified notation, where a simset $\mathscr{S}_p$ may denote *any* simset of a leaf Σ_p, including the

6 A simply connected boundary may not admit holes, but being codimension one in $D = 4$, one may still have missing events. A physically acceptable boundary should be also free of such pathologies.

whole leaf Σ_p itself. In turn, $\mathscr{B}$ could be a causal boundary defined as an appropriate 'causal extensions' of the boundary $\partial\mathscr{S}_p \subset \Sigma_p$, or it could denote some suitable subset of $\partial\mathscr{M}$, or it may even be some consistent combination of both kinds of boundaries. In all these cases however $\mathscr{B}$ must satisfy condition (i) of Definition 2. The developments of a whole leaf Σ_p with respect to some suitable boundary $\mathscr{B} \subseteq \partial\mathscr{M}$ will sometimes be denoted as $D^{\pm}(\Sigma_p, \mathscr{B})$ for concreteness. Thanks to our unified notation, however, most of the claims and conclusions in what follows will apply for all situations.

A first property of the domains of dependence defined above is that

$$D^{+}(\mathscr{S}_p, \mathscr{B}) \subseteq J^{+}(\Sigma_p) \,, \qquad D^{-}(\mathscr{S}_p, \mathscr{B}) \subseteq J^{-}(\Sigma_p) \,. \tag{4.26}$$

Furthermore by the causality relations in (4.12) and the definitions in (4.15) for the closure of the sets $J^{\pm}(\Sigma_p)$, we have

$$D^{+}(\mathscr{S}_p, \mathscr{B}) \cap \overline{J^{-}(\Sigma_p)} = \emptyset \,, \qquad D^{-}(\mathscr{S}_p, \mathscr{B}) \cap \overline{J^{+}(\Sigma_p)} = \emptyset \,. \tag{4.27}$$

These relations between the developments and causal past/future of Σ_p will be useful below.

Guided by intuitions from general relativity, one would now expect that an event in $J^{+}(\Sigma_p)$ will cease to be in the domain $D^{+}(\mathscr{S}_p, \mathscr{B})$ if a future Cauchy horizon forms; indeed a future Cauchy horizon is expected to mark the end of a domain of dependence, beyond which prediction is no longer possible. The reasons why such a Cauchy horizon may form could be varied and are not among our concerns here. Rather, we are interested in the most general properties of this type of horizons. Appropriate redefinitions will as usual lead to the notion of a *past Cauchy horizon*.

Towards a formal definition of Cauchy horizons in the present context, let us consider the closures of the domains of dependence in $\mathscr{M}$, to be denoted by $\overline{D^{+}(\mathscr{S}_p, \mathscr{B})}$ and $\overline{D^{-}(\mathscr{S}_p, \mathscr{B})}$, respectively. Note that, by using (4.26),

$$\overline{D^{+}(\mathscr{S}_p, \mathscr{B})} \subseteq \overline{J^{+}(\Sigma_p)} \,, \qquad \overline{D^{-}(\mathscr{S}_p, \mathscr{B})} \subseteq \overline{J^{-}(\Sigma_p)} \,. \tag{4.28}$$

Naturally, the closures of the domains of dependence — considered in $\mathscr{M}$ — will trivially contain the simset $\mathscr{S}_p$ as well as the relevant part of the boundaries $\mathscr{B}$. Therefore, as an indicator of the presence of any non-trivial boundary events in the spacetime which would mark a true end of the development as noted above, let us introduce the following notation for the boundary events of the domains that are not part of the 'trivial boundaries'

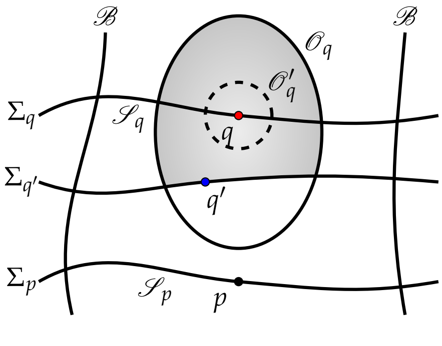
$$\begin{aligned} H^+(\mathscr{S}_p, \mathscr{B}) &\equiv \partial D^+(\mathscr{S}_p, \mathscr{B}) \setminus (\mathscr{S}_p \cup \mathscr{B}') \ , \\ H^-(\mathscr{S}_p, \mathscr{B}) &\equiv \partial D^-(\mathscr{S}_p, \mathscr{B}) \setminus (\mathscr{S}_p \cup \mathscr{B}') \ , \end{aligned} \tag{4.29}$$

where $\mathscr{B}' \subseteq \mathscr{B}$ denotes the part of $\mathscr{B}$ that is not in $\partial\mathscr{M}$. From (4.28), we then have $H^+(\mathscr{S}_p, \mathscr{B}) \subseteq J^+(\Sigma_p)$ and $H^-(\mathscr{S}_p, \mathscr{B}) \subseteq J^-(\Sigma_p)$ as one would expect intuitively as well. We will henceforth define $H^+(\mathscr{S}_p, \mathscr{B})$ as the *future Cauchy horizon* of the future domain $D^+(\mathscr{S}_p, \mathscr{B})$, and likewise define $H^-(\mathscr{S}_p, \mathscr{B})$ as the *past Cauchy horizon* of the past domain $D^-(\mathscr{S}_p, \mathscr{B})$. When the developments of entire leaves are under consideration, the appropriate definitions of the Cauchy horizons $H^\pm(\Sigma_p, \mathscr{B})$ of the domains $D^\pm(\Sigma_p, \mathscr{B})$ should be

$$\begin{aligned} H^+(\Sigma_p, \mathscr{B}) &\equiv \partial D^+(\Sigma_p, \mathscr{B}) \setminus \Sigma_p \ , \\ H^-(\Sigma_p, \mathscr{B}) &\equiv \partial D^-(\Sigma_p, \mathscr{B}) \setminus \Sigma_p \ , \end{aligned} \tag{4.30}$$

instead of (4.29). As before, we have $H^+(\Sigma_p, \mathscr{B}) \subseteq J^+(\Sigma_p)$ and $H^-(\Sigma_p, \mathscr{B}) \subseteq J^-(\Sigma_p)$, following from the analogues of (4.26) and (4.28).

We will now show that such boundary events form part or whole of a leaf. However, before doing so, a few remarks are necessary to clarify our presentation. In general relativity it is natural to begin with the conventional definition of the Cauchy horizon (Wald, 1984, Equation 8.3.3) before establishing that it is a null hypersurface (Wald, 1984, Theorem 8.3.5) and a boundary of the domain of dependence (Wald, 1984, Proposition 8.3.6). Here instead, it is more natural to start with the definitions of Cauchy horizons as non-trivial boundary events of the corresponding domains of dependence and then proceed to establish that they form a leaf. The domains of influence and dependence are manifestly different concepts in general relativity, while as to be proven below, the domains of complete leaves are identical with their



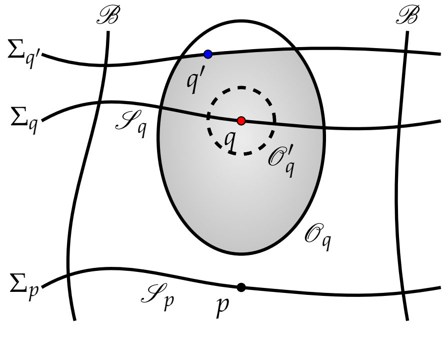


Figure 5: Figures for the proof of Theorem 1 case 1.

past/future in the absence of any Cauchy horizons. Blindly following the standard presentation of general relativity (along the lines of Wald, 1984) would seemingly prevent us from emphasizing the central role played by boundaries and the crucial differences stemming from the causal structure in the present context. Nevertheless, we will show below that $H^{\pm}(\Sigma_p, \mathscr{B})$ as defined above in (4.30) also satisfy the conventional definition of Cauchy horizons.

With these clarifications out of the way, we may now formulate the following Theorem.

Theorem 1. *$H^+(\mathscr{S}_p, \mathscr{B})$, if non-empty, is a simset with empty future development, i.e. $D^+(H^+(\mathscr{S}_p, \mathscr{B}), \mathscr{B}) = \emptyset$. Likewise, $H^-(\mathscr{S}_p, \mathscr{B})$, if non-empty, is a simset with empty past development, i.e. $D^-(H^-(\mathscr{S}_p, \mathscr{B}), \mathscr{B}) = \emptyset$.*

Proof. Consider the case for the future domain first. $H^+(\mathscr{S}_p, \mathscr{B})$ is non-empty by assumption, so there is at least one event $q \in H^+(\mathscr{S}_p, \mathscr{B})$. Furthermore, $H^+(\mathscr{S}_p, \mathscr{B})$ is part of a boundary by definition. Therefore, every open neighbourhood $\mathscr{O}_q$ of $q \in H^+(\mathscr{S}_p, \mathscr{B})$ must contain events that are in the development $D^+(\mathscr{S}_p, \mathscr{B})$ as well as events that are not in the development $D^+(\mathscr{S}_p, \mathscr{B})$; in other words, the neighbourhood $\mathscr{O}_q$ must satisfy $\mathscr{O}_q \cap D^+(\mathscr{S}_p, \mathscr{B}) \neq \emptyset$ as well as $\mathscr{O}_q \cap D^+(\mathscr{S}_p, \mathscr{B})^c \neq \emptyset$.

The rest of the proof calls for separate analyses of the following two cases:

Case 1: The leaf Σ_q satisfies condition (i) of Definition 2. We will denote the simset that defines the interior of the closure by $\mathscr{S}_q$ (see Figure 5A).

Now, suppose some leaf $\Sigma_{q'} \subset J^-(\Sigma_q) \cap J^+(\Sigma_p)$ failed to satisfy condition (i) of Definition 2. Then events in $\Sigma_{q'} \cup J^+(\Sigma_{q'})$ cannot be in the development $D^+(\mathscr{S}_p, \mathscr{B})$ by definition, allowing us to construct an open neighbourhood of q in $J^-(\Sigma_q) \cap J^+(\Sigma_{q'})$ not containing any event in the said development either. But this would contradict the fact that q is a boundary event. Therefore, every leaf in $J^-(\Sigma_q) \cap J^+(\Sigma_p)$ must satisfy condition (i) of Definition 2. Note that leaves in $J^+(\Sigma_q)$ may or may not satisfy this condition, without any loss in generality.

Now, pick an open neighbourhood $\mathscr{O}_q \subset J^+(\Sigma_p)$ of q such that $\mathscr{O}_q \cap \Sigma_q$ is contained in $\mathscr{S}_q$ (or equal to the latter), and choose an event $q' \in \mathscr{O}_q \cap J^-(\Sigma_q)$ (e.g. the blue point in Figure 5A). If q' is not in the development $D^+(\mathscr{S}_p, \mathscr{B})$, then neither is any event in $\mathscr{O}_q \cap J^+(\Sigma_{q'})$ (the shaded region in Figure 5A), since at least one past inextendible causal curve from each event in $\mathscr{O}_q \cap J^+(\Sigma_{q'})$ must pass through q' and therefore cannot be extended to $\mathscr{S}_p$ or $\mathscr{B}$, thereby violating condition (ii) of Definition 2. But then one will always be able to construct a 'smaller' neighbourhood $\mathscr{O}'_q \subset \mathscr{O}_q \cap J^+(\Sigma_{q'})$ (e.g. the region inside the dotted circle in Figure 5A) such that $\mathscr{O}'_q$ is also not contained in the development $D^+(\mathscr{S}_p, \mathscr{B})$, i.e. $\mathscr{O}'_q \cap D^+(\mathscr{S}_p, \mathscr{B}) = \emptyset$. This is a contradiction of the starting assumption that q is a boundary event. Therefore, $\mathscr{O}_q \cap J^-(\Sigma_q)$ must consist of events only belonging to the development $D^+(\mathscr{S}_p, \mathscr{B})$, i.e. $\mathscr{O}_q \cap J^-(\Sigma_q) \subseteq D^+(\mathscr{S}_p, \mathscr{B})$.

Similarly, $\mathscr{O}_q \cap J^+(\Sigma_q)$ cannot contain any event from the development $D^+(\mathscr{S}_p, \mathscr{B})$. For otherwise, if $q' \in D^+(\mathscr{S}_p, \mathscr{B})$ for some $q' \in J^+(\Sigma_q)$ (e.g. the blue point in Figure 5B) – which presupposes that the leaf $\Sigma_{q'}$ satisfies condition (i) of Definition 2 – then so must be every event in $\mathscr{O}_q \cap J^-(\Sigma_{q'})$ (the shaded region in Figure 5B). This, in turn, should again allow us to construct a 'smaller' neighbourhood $\mathscr{O}'_q \subset \mathscr{O}_q \cap J^-(\Sigma_{q'})$ (e.g. the region inside the dotted circle in Figure 5B) such that $\mathscr{O}'_q$ consists of events which only belong to the development $D^+(\mathscr{S}_p, \mathscr{B})$, i.e. $\mathscr{O}'_q \cap D^+(\mathscr{S}_p, \mathscr{B})^c = \emptyset$. This is again a contradiction of the starting assumption of q being a boundary event. Therefore, $\mathscr{O}_q \cap J^+(\Sigma_q)$ must consist of events which do not belong to the development $D^+(\mathscr{S}_p, \mathscr{B})$, i.e. $\mathscr{O}_q \cap J^+(\Sigma_q) \subseteq D^+(\mathscr{S}_p, \mathscr{B})^c$.

Next, consider an event q'' on the simset $\mathscr{O}_q \cap \Sigma_q$ which is different from q. By the results just derived, every open neighbourhood $\mathscr{O}_{q''}$ must contain events both in $D^+(\mathscr{S}_p, \mathscr{B})$ and its complement. Hence q'' is a 'boundary event', i.e. $q'' \in H^+(\mathscr{S}_p, \mathscr{B})$. But since this is true for every event in $\mathscr{O}_q \cap \Sigma_q$, the entire simset $\mathscr{O}_q \cap \Sigma_q$ must consist only of such 'boundary events' so that $\mathscr{O}_q \cap \Sigma_q \subseteq H^+(\mathscr{S}_p, \mathscr{B})$. However, no particular assumption was made about the open neighbourhood $\mathscr{O}_q$ here, except that it is entirely to the future of Σ_p and that $\mathscr{O}_q \cap \Sigma_q \subseteq \mathscr{S}_q$. Consequently, any event not in $\mathscr{S}_q$ is not a non-trivial 'boundary event' of $D^+(\mathscr{S}_p, \mathscr{B})$ in the sense of (4.29). In other words, $H^+(\mathscr{S}_p, \mathscr{B})$ is composed entirely of the events on the simset $\mathscr{S}_q$ and only these. This proves the first part of the proposition for $D^+(\mathscr{S}_p, \mathscr{B})$, namely $H^+(\mathscr{S}_p, \mathscr{B})$ is a simset; in fact $H^+(\mathscr{S}_p, \mathscr{B}) = \mathscr{S}_q$ by the above proof.

Now suppose that the development of $H^+(\mathscr{S}_p, \mathscr{B})$ is non-empty, i.e. there exists some event $r \in J^+(H^+(\mathscr{S}_p, \mathscr{B}))$ such that $r \in D^+(H^+(\mathscr{S}_p, \mathscr{B}), \mathscr{B})$. This necessarily requires the leaf Σ_r to satisfy condition (i) of Definition 2. Furthermore, since every past inextendible curve from r must either intersect $H^+(\mathscr{S}_p, \mathscr{B})$ or reach $\mathscr{B}$ by Definition 2, by suitably extending such curves in the past, we may conclude that $r \in D^+(\mathscr{S}_p, \mathscr{B})$ as well. This, however, is a contradiction of the fact that $H^+(\mathscr{S}_p, \mathscr{B})$ is a boundary of the development $D^+(\mathscr{S}_p, \mathscr{B})$. Thus $D^+(H^+(\mathscr{S}_p, \mathscr{B}), \mathscr{B}) = \emptyset$. This completes the proof of the theorem for this case for the future domain of dependence $D^+(\mathscr{S}_p, \mathscr{B})$.

In a similar fashion, one may prove for this case that $H^-(\mathscr{S}_p, \mathscr{B})$, if non-empty, is a simset as well and that $D^-(H^-(\mathscr{S}_p, \mathscr{B}), \mathscr{B}) = \emptyset$.

Case 2: The leaf Σ_q does *not* satisfies condition (i) of Definition 2. Hence no event in $\Sigma_q \cup J^+(\Sigma_q)$ can be contained in the development $D^+(\mathscr{S}_p, \mathscr{B})$.

Now, suppose $H^+(\mathscr{S}_p, \mathscr{B})$ contains an event q' which is also in $J^+(\Sigma_q)$. From our preceding conclusions $q' \notin D^+(\mathscr{S}_p, \mathscr{B})$, and in fact, there must then exist an open neighbourhood $\mathscr{O}_{q'} \subset J^+(\Sigma_q)$ which does not contain any event in the development $D^+(\mathscr{S}_p, \mathscr{B})$. Therefore q' cannot be both in $H^+(\mathscr{S}_p, \mathscr{B})$ (i.e. be a boundary event) and in $J^+(\Sigma_q)$.

Assume instead $H^+(\mathscr{S}_p, \mathscr{B})$ contains an event q'' which is also in $J^-(\Sigma_q) \cap J^+(\Sigma_p)$. If $\Sigma_{q''}$ did *not* satisfy condition (i) of Definition 2, interchanging q and q'' and rerunning the argument as above would imply that q is not a boundary event – a contradiction. If instead $\Sigma_{q''}$ *did* satisfy condition (i) then the proof for Case 1 above applies to q'' and this implies that *every* other event in $H^+(\mathscr{S}_p, \mathscr{B})$ should also be in $\Sigma_{q''}$. This leads to a yet another contradiction, as $q \in H^+(\mathscr{S}_p, \mathscr{B})$ but not in $\Sigma_{q''}$ by assumption. In conclusion, if Σ_q does not satisfies condition (i) then any other event in $H^+(\mathscr{S}_p, \mathscr{B})$ is also in Σ_q. Hence $H^+(\mathscr{S}_p, \mathscr{B})$ is a simset. Moreover, since $\Sigma_q \supseteq H^+(\mathscr{S}_p, \mathscr{B})$ does not satisfies condition (i) we have $D^+(H^+(\mathscr{S}_p, \mathscr{B}), \mathscr{B}) = \emptyset$ by Definition 2.

In a similar fashion, one may prove for this case that $H^-(\mathscr{S}_p, \mathscr{B})$, if non-empty, is a simset as well and that $D^-(H^-(\mathscr{S}_p, \mathscr{B}), \mathscr{B}) = \emptyset$. □

While studying the domains of dependence of proper simsets with respect to some 'artificial' boundaries may be interesting in some situations, the domains of dependence of complete leaves in a part of the spacetime admitting an asymptotic region are far more important for our purposes. In the remainder of this Section, we therefore focus our attention exclusively on this type of spacetimes.

Let $\Sigma_p \subset \langle\langle \mathscr{M} \rangle\rangle$ be a leaf admitting a trivially foliated flat end. Its domains of dependence with respect to the leaf Σ_p and some boundary $\mathscr{B} \subseteq \partial\mathscr{M}$ [recall (4.23)] will be denoted by $D^{\pm}(\Sigma_p, \mathscr{B})$; in particular, $\mathscr{I} \subseteq \mathscr{B}$. According to Theorem 1 the non-trivial boundaries $H^{\pm}(\Sigma_p, \mathscr{B})$, if non-empty, constitutes leaves of the foliation and act as Cauchy horizons. From the proof of Theorem 1, along with (4.27),[7] we have

$$\begin{aligned} D^+(\Sigma_p,\mathscr{B}) &= J^-(H^+(\Sigma_p,\mathscr{B}))\cap J^+(\Sigma_p)\ , \\ D^+(\Sigma_p,\mathscr{B})^c &= \overline{J^+(H^+(\Sigma_p,\mathscr{B}))}\cup\overline{J^-(\Sigma_p)}\ , \\ D^-(\Sigma_p,\mathscr{B}) &= J^+(H^-(\Sigma_p,\mathscr{B}))\cap J^-(\Sigma_p)\ , \\ D^-(\Sigma_p,\mathscr{B})^c &= \overline{J^-(H^-(\Sigma_p,\mathscr{B}))}\cup\overline{J^+(\Sigma_p)}\ . \end{aligned} \tag{4.31}$$

Hence in particular, $D^\pm(\Sigma_p,\mathscr{B})$ are open sets, and, therefore, analogous to the relevant results in Wald (1984, Lemma 8.3.3). It is also easy to conclude from the above that

$$\begin{aligned} J^-(H^+(\Sigma_p,\mathscr{B})) &= J^-(D^+(\Sigma_p,\mathscr{B})) = D^+(\Sigma_p,\mathscr{B})\cup\overline{J^-(\Sigma_p)}\ , \\ J^+(H^-(\Sigma_p,\mathscr{B})) &= J^+(D^-(\Sigma_p,\mathscr{B})) = D^-(\Sigma_p,\mathscr{B})\cup\overline{J^+(\Sigma_p)}\ . \end{aligned} \tag{4.32}$$

Furthermore, the closures of the domains *in* $\mathscr{M}$ are given as[8]

$$\begin{aligned} \overline{D^+(\Sigma_p,\mathscr{B})} &= H^+(\Sigma_p,\mathscr{B})\cup D^+(\Sigma_p,\mathscr{B})\cup\Sigma_p\ , \\ \overline{D^-(\Sigma_p,\mathscr{B})} &= H^-(\Sigma_p,\mathscr{B})\cup\Sigma_p\cup D^-(\Sigma_p,\mathscr{B})\ . \end{aligned} \tag{4.33}$$

From the above relations, we then also have formal agreement with the standard definitions of Cauchy horizons in general relativity as follows

$$\begin{aligned} H^+(\Sigma_p,\mathscr{B}) &= \overline{D^+(\Sigma_p,\mathscr{B})}\setminus J^-(D^+(\Sigma_p,\mathscr{B}))\ , \\ H^-(\Sigma_p,\mathscr{B}) &= \overline{D^-(\Sigma_p,\mathscr{B})}\setminus J^+(D^-(\Sigma_p,\mathscr{B}))\ . \end{aligned} \tag{4.34}$$

One may now define the *full domain of dependence* $D(\Sigma_p,\mathscr{B})$ of the leaf Σ_p as the union of the leaf itself along with its past and future domains of dependence (note that the corresponding definition of general relativity only involves the union of the past and future developments)

$$\begin{aligned} D(\Sigma_p,\mathscr{B}) &\equiv D^+(\Sigma_p,\mathscr{B})\cup\Sigma_p\cup D^-(\Sigma_p,\mathscr{B})\ , \\ &= J^-(H^+(\Sigma_p,\mathscr{B}))\cap J^+(H^-(\Sigma_p,\mathscr{B}))\ , \end{aligned} \tag{4.35}$$

7 And by the trivial fact that if three sets X, Y and Z satisfy $X\subseteq Z$, $Y\subseteq Z^c$ and $X\cup Y=\mathscr{M}$, then $X=Z$ and $Y=Z^c$.

8 The closures in the conformal extension should also include the boundary $\mathscr{B}$.

where the second equality follows from (4.31). Hence as expected, the full domain is an open set. The closure of the full domain $D(\Sigma_p, \mathscr{B})$ *in* $\mathscr{M}$[9] is then given by

$$\overline{D(\Sigma_p, \mathscr{B})} = H^+(\Sigma_p, \mathscr{B}) \cup D(\Sigma_p, \mathscr{B}) \cup H^-(\Sigma_p, \mathscr{B}) \ . \tag{4.36}$$

So, if we define the *full Cauchy horizon* of Σ_p, to be denoted by $H(\Sigma_p, \mathscr{B})$, as

$$H(\Sigma_p, \mathscr{B}) \equiv H^+(\Sigma_p, \mathscr{B}) \cup H^-(\Sigma_p, \mathscr{B}) \ , \tag{4.37}$$

we find that the boundary of $D(\Sigma_p, \mathscr{B})$ *in* $\mathscr{M}$[10] is nothing but $H(\Sigma_p, \mathscr{B})$ as expected (compare with Wald, 1984, Proposition 8.3.6)

$$\partial D(\Sigma_p, \mathscr{B}) \equiv \overline{D(\Sigma_p, \mathscr{B})} \setminus D(\Sigma_p, \mathscr{B}) = H(\Sigma_p, \mathscr{B}) \ . \tag{4.38}$$

In this way, the future domain of dependence $D^+(\Sigma_p, \mathscr{B})$ of the leaf Σ_p, the corresponding future Cauchy horizon $H^+(\Sigma_p, \mathscr{B})$, as well as their 'past' analogues' $D^-(\Sigma_p, \mathscr{B})$ and $H^-(\Sigma_p, \mathscr{B})$ share many of the features of the corresponding notions of general relativity.

As in general relativity, a leaf Σ_p for which $D(\Sigma_p, \mathscr{B}) = \mathscr{M}$ will be called a *Cauchy surface* and a spacetime that possesses a Cauchy surface will be regarded as *globally hyperbolic*. From (4.35) it is obvious that for a leaf Σ_p with a past and/or future Cauchy horizon, $D(\Sigma_p, \mathscr{B}) \neq \mathscr{M}$. On the other hand, in order to determine when the full domain can be the full spacetime, we may resort to the following Theorem.

Theorem 2. *If the future development of a leaf Σ_p is non-empty yet no future Cauchy horizon forms, then the causal future of the leaf is identical with its future domain of dependence, i.e.*

$$D^+(\Sigma_p, \mathscr{B}) \neq \emptyset \ , \ H^+(\Sigma_p, \mathscr{B}) = \emptyset \quad \Rightarrow \quad D^+(\Sigma_p, \mathscr{B}) = J^+(\Sigma_p) \ .$$

Likewise, if the past development of a leaf Σ_p is non-empty yet no past Cauchy horizon forms, then the causal past of the leaf is identical with its past domain of dependence, i.e.

$$D^-(\Sigma_p, \mathscr{B}) \neq \emptyset \ , \ H^-(\Sigma_p, \mathscr{B}) = \emptyset \quad \Rightarrow \quad D^-(\Sigma_p, \mathscr{B}) = J^-(\Sigma_p) \ .$$

9 The closure in the conformal extension should also contain $\mathscr{B}$.
10 The boundary in the conformal extension should also contain $\mathscr{B}$.

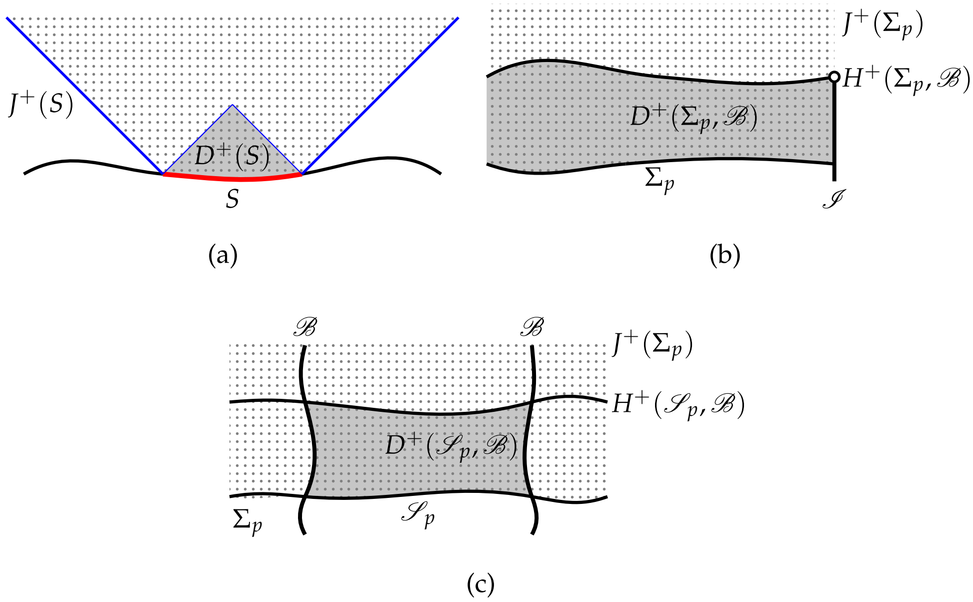


Figure 6: Difference between the notions of Cauchy development and Cauchy horizons in locally Lorentz invariant theories (A) and theories with a preferred foliation, both for boundary at infinity (B) and in the bulk (C).

Proof. Consider the case with the future development first. By our assumptions, every past inextendible causal curve through every $q \in J^+(\Sigma_p)$ must intersect Σ_p or reach $\mathscr{B}$, implying $J^+(\Sigma_p) \subseteq D^+(\Sigma_p, \mathscr{B})$. Appealing to (4.26) after setting $\mathscr{S}_p = \Sigma_p$ in that equation, we then have $D^+(\Sigma_p, \mathscr{B}) = J^+(\Sigma_p)$ under the relevant assumptions. An obviously analogous proof exists for the past domain under similar assumptions. $\square$

It can be worth comparing and contrasting the results of Theorem 2 along with those in (4.13), as they convey the peculiarities of Lorentz violating causality in a very succinct yet effective manner (see Figure 6 (A) and (B) for a comparison). One may also contrast the above with the consequences of Theorem 1 as summarized in (4.31). We should emphasize in this regard that both the requirements of non-empty developments and empty Cauchy horizons are necessary in the statements of Theorem 2. In particular, a Cauchy horizon provides an example of a leaf which does not itself have its own Cauchy horizons, yet its non-empty causal past and future do not agree with its respective past and future developments, both of the latter being empty. Finally, as a trivial consequence of the Theorems 1, 2, and the relation (4.14), we have that *a leaf Σ_p with a non-empty development is a Cauchy hypersurface if and only if its full Cauchy horizon $H(\Sigma_p, \mathscr{B})$ is empty.* This is analogous to the Corollary to Wald (1984, Proposition 8.3.6).

Even if a leaf Σ_p possesses past and/or future Cauchy horizons, the data provided on it are capable of determining the region in the full development $D(\Sigma_p, \mathscr{B})$. Therefore, one may regard such a leaf as a *partial Cauchy surface*. In the same vein, one may regard the full development (4.35) as globally hyperbolic, since it is completely built up with the data provided on any such partial Cauchy surface.

♦

In the present Chapter we have established the fundamentals for studying the causal structure of a spacetime with a preferred foliation. We have provided the reader with the necessary definition for understanding past and future in this type of spacetimes, and we have discussed the ways we have to construct the asymptotics for such spacetimes. To finish, we have studied in detail the concepts of causal development and of Cauchy horizons in the present context.

Having these basic concepts, it is time finally to try and define in a rigorous manner the concepts of black holes and universal horizons in theories that violate Lorentz symmetry through the presence of a preferred foliation. This is the plan for next Chapter.

5

BLACK HOLES

Black holes are among the most interesting objects the Universe provides us the possibility to study. Since the discovery of the first black hole solution in general relativity (Schwarzschild, 1916), such objects have been studied in quite some detail by theoretical physicists over the years.

In a way, the concept of a black hole is a relatively simple one. The "cartoon definition" of such objects, which is relatively accurate despite its naïveté, is that of an object with a mass density high enough that even light cannot escape its gravitational field. This is quite clearly a simplistic version of the definition of black hole, but gives nevertheless the right taste of what these objects are.

A more rigorous definition can be obtained by specifying better what we mean with 'escaping the gravitational field'. In fact, once we have at our disposal the definitions for the asymptotic regions of a spacetime, we can rephrase the definition of black hole in GR as the portion of spacetime which is not connected via null curves to null infinity (see for more details Hawking and Ellis, 1973). Even though this last definition might seem very different from the intuitive one provided just above, they are actually almost the same thing. Indeed, once we accept that escaping the black hole means reaching 'infinity' along a causally allowed direction, the similarity becomes apparent since the part of spacetime disconnected from infinity is exactly the one where no causal signal can reach infinity.

As we mentioned before, on the other hand, in general relativity the existence of black holes, and even more so their definition, hinges on the causal structure that follows from Lorentz symmetry on matter fields and from the local flatness theorem. It's a fundamental question therefore whether, in the absence of such symmetry, black holes will still be able to form; if they do, it is even more important to study their characteristics, since many hopes to observe signatures of Lorentz violations lie in the strong gravity regime.

Even when we break Lorentz symmetry and the causal structure is modified in a radical way, as we mentioned in the previous Chapter, black hole solutions surprisingly do exist. In this new black hole solutions, the event horizon is replaced by the so called *universal horizon* (Barausse et al., 2011; Blas and Sibiryakov, 2011), whose property is to trap any mode independently on the propagation speed. Such black hole solutions have been already found in restricted Lorentz violating settings, such as spherical symmetry (Eling and Jacobson, 2006; Barausse et al., 2011) and slowly-rotating background, both in lower dimensions (Sotiriou et al., 2014) and $1+3$-dimensions (Barausse and Sotiriou, 2012, 2013a,b; Barausse et al., 2016). In addition, these solutions were mostly found by numerically solving the equations of Hořava gravity and Einstein-Æther theory; there are only few analytic solutions for this type of theories and all those are found in symmetry restricted scenarios for various asymptotics (Bhattacharyya and Mattingly, 2014). There is therefore no generic analytical solution that allows one to study the features of this kind of horizons.

In the present Chapter, we will try to give a definition, as general as possible, for black holes and universal horizons without resorting to any symmetries or other restrictions. Having found the generic definition, we will then try to study some restricted scenarios — mainly stationary and more symmetric spacetimes — towards a local characterisation of black holes, and with the aim of comparing the new solutions to the ones traditionally found in general relativity.

5.1 KNOWN SOLUTIONS

Despite the fact of not having available a rigorous definition of black hole in Lorentz violating settings, some solutions in restricted setting are nevertheless available in the literature mentioned above. It could then be interesting to discuss the results found within these solutions, so to have some intuition to draw on when we will give rigorous definitions of black holes later on in the Chapter. Since it's far from the main goal of this work, we will avoid going into the details of how the particular solutions were obtained and we will instead simply discuss the characteristics of the results obtained, concentrating mainly on how the universal horizon is defined.

5.1.1 *The universal horizon*

The first thing we wish to understand is how the concept of universal horizon came about in Barausse et al. (2011) — and later on in Blas and Sibiryakov (2011). These solutions were found in a spherically symmetry, asymptotically flat spacetime, where the æther is forced to be hypersurface-orthogonal by the symmetry and hence the solutions of Einstein-Æther theory and of Hořava gravity coincide.

The main result in this case was obtained by computing the angle between the æther — which indicates unambiguously the direction of time (T) evolution — and the normal to the hypersurfaces with constant radius (Barausse et al., 2011). This angle, sometimes called the *boost angle* is given by

$$\theta_r = \operatorname{arccosh}\left(u \cdot \frac{\mathrm{d}r}{\sqrt{g^{rr}}}\right) . \tag{5.1}$$

The crucial feature here is that the boost angle vanishes for a finite value of the radial variable r, thus indicating that one particular leaf of the constant-T foliation coincides with one particular leaf of the constant-r foliation. This tells us that any leaf in the future of such particular leaf will never be able to reach — or connect to — infinity, since two leaves of the foliation cannot intersect (remember the properties of an ordered foliation discussed in Section 4.1 of Chapter 4). This particular leaf therefore represents a causal barrier that cannot be crossed by any signal moving to the future, irrespectively on the speed of such signal. Precisely this leaf, which we will denote as $\Sigma_{\rm H}$ for reasons to become clear in the following, is what constitutes the *universal horizon* — dubbed this way in Barausse et al. (2011); Blas and Sibiryakov (2011) precisely for the property of being able to trap modes of any speed.

5.1.2 *Universal horizons in rotating spacetimes*

In the rotating case, where the spacetime is axysymmetric instead than spherically symmetric, the æther is not forced to be hypersurface orthogonal and the solutions of Hořava gravity and of Einstein-Æther theory will be different.

In particular no universal horizons seem to exist in Einstein-Æther theory, due to the fact that no global preferred-time foliation exists in the first place (Barausse and Sotiriou, 2013b).

The case is different for Hořava gravity where, as mentioned before, a global preferred time foliation always necessarily exists. In this case an universal horizon does indeed exist. In the 4-dimensional case, the only solutions known up to now are obtained in the slow rotation limit. since the effects of slow rotation contribute only as perturbations though, the universal horizon coincides with the one found in the spherically symmetric case (Barausse and Sotiriou, 2013b).

Interestingly enough though, in the 3-dimensional case black holes were shown to exists with various asymptotics and in the generic rotating case (Sotiriou et al., 2014). Some of the solutions found do also exibit the presence of an universal horizon.

5.2 EVENT HORIZONS, UNIVERSAL HORIZONS, BLACK HOLES

At this point, having discussed the black hole solutions available in the literature, we are finally ready to give a rigorous definition for black holes in presence of Lorentz violations. This definition will rely heavily on the results of Chapter 4 on the causal structure and asymptotics of the spacetime. We will hence, in line with what done in said Chapter, study black holes in spacetimes with a preferred foliation. As we discussed before, these spacetimes represent indeed the right setting for answering the questions we have in mind — mainly related to Hořava gravity — without the need to be too theory-specific. For this reason, the definitions we will discuss in the present Chapter will hold in any theory admitting a spacetime with the structure we are considering.

Once we obtain the definition of black hole in the general case, we will move on to discuss the characteristics of the horizons defined through such solution, and we will try to give a more "practical" way to find the locus of the horizon. Finally, at the end of the Chapter, we will discuss the properties of some particular cases with restricted symmetries, which can be readily compared to the solutions already available and summarised above.

5.2.1 *Event horizons and black holes*

In this Section, we finally turn our attention to the concepts of black and white holes and the corresponding notions of event horizons. As in general relativity, we would like to define a black hole as a region of the spacetime from which causal influences can 'never escape'. An event horizon, by definition, should then mark the boundary of such region. In the present context however, causal influences can propagate arbitrarily fast and are not restricted to remain within the propagation cone of any speed-c metric (4.8). As such, the following definitions of black/white holes and their event horizons should appropriately reflect the difference from the corresponding general relativistic concepts.

In order to formalize the definition of a black hole in the most general setting, we need to properly clarify the notion of 'never escaping' a region of spacetime. To that end, consider an event $p \in \mathscr{M}$ which is *not* inside a black hole. Then, an arbitrarily fast excitation leaving p will move far away towards some asymptotic region. In Section 4.3, we formalized the notion of such an asymptotic region as some suitably chosen neighborhood of the boundary at infinity $\mathscr{I}$. Therefore, a rigorous way to interpret the idea of 'escaping' — as in the definition of a black hole — will be to reach any arbitrary open neighbourhood of $\mathscr{I}$ along a causally allowed direction. An event $p \in \mathscr{M}$ will *not* be inside a black hole region if one can find a future directed causal curve through p which enters any such neighbourhood of $\mathscr{I}$.

The definition of the asymptotic boundary $\mathscr{I}$ given in Section 4.3 comes alongside that of an open region $\langle\langle\mathscr{M}\rangle\rangle \subseteq \mathscr{M}$ with the property that every leaf $\Sigma_p \subset \langle\langle\mathscr{M}\rangle\rangle$ admits a point at infinity $\mathfrak{i}_p$ (see (4.22)) in the conformal extension of the spacetime. In particular, by invoking the connectedness of $\langle\langle\mathscr{M}\rangle\rangle$ and that of every leaf in it, one may construct an acausal curve through any event $q \in \Sigma_p \subset \langle\langle\mathscr{M}\rangle\rangle$ which remains entirely confined in Σ_p and enters any open neighbourhood of $\mathfrak{i}_p$. Adding an appropriate component along the æther to the tangent to any such curve, one may then construct a curve $\lambda(\tau) \subset \langle\langle\mathscr{M}\rangle\rangle \cup \mathscr{I}$ for $\tau \in [0,1]$ through any $p \in \langle\langle\mathscr{M}\rangle\rangle$, with $\lambda(0) = p$ and $\lambda(1) \in \mathscr{I}$, such that the curve is future directed causal in $\langle\langle\mathscr{M}\rangle\rangle$. In other words, $\langle\langle\mathscr{M}\rangle\rangle$ consists of events from which there always exists at least one future directed causal curve which enters any arbitrary neighbourhood of $\mathscr{I}$. Similar considerations show that $\langle\langle\mathscr{M}\rangle\rangle$ also consists of events from which there always exists at least one past directed causal curve which enters any arbitrary neighbourhood of $\mathscr{I}$. Thus the open region $\langle\langle\mathscr{M}\rangle\rangle$ seemingly has the right properties to be interpreted as (at least part of) the 'outside' region of a black/white hole spacetime.

Now, from the definitions given in (4.9), we have

$$\begin{aligned} J^-(\mathscr{I}) &= \bigcup_{p \in \langle\langle \mathscr{M} \rangle\rangle} J^-(\mathrm{i}_p) = \bigcup_{p \in \langle\langle \mathscr{M} \rangle\rangle} J^-(\Sigma_p) = J^-(\langle\langle \mathscr{M} \rangle\rangle) \,, \\ J^+(\mathscr{I}) &= \bigcup_{p \in \langle\langle \mathscr{M} \rangle\rangle} J^+(\mathrm{i}_p) = \bigcup_{p \in \langle\langle \mathscr{M} \rangle\rangle} J^+(\Sigma_p) = J^+(\langle\langle \mathscr{M} \rangle\rangle) \,. \end{aligned} \tag{5.2}$$

In both cases above, the second equality follows from (4.13), while the final equality invokes the definitions of past and future sets in (4.9) once more. Since $\langle\langle \mathscr{M} \rangle\rangle$ is open by definition and the past and future sets are open — as argued previously — one has $\langle\langle \mathscr{M} \rangle\rangle \subseteq J^\pm(\langle\langle \mathscr{M} \rangle\rangle)$, and hence $\langle\langle \mathscr{M} \rangle\rangle \subseteq J^\pm(\mathscr{I})$ according to (5.2). In fact, in a trivially foliated flat spacetime $J^-(\mathscr{I}) = J^+(\mathscr{I}) = \langle\langle \mathscr{M} \rangle\rangle = \mathscr{M}$. More generally however, there could be parts of $J^\pm(\mathscr{I})$ which are outside $\langle\langle \mathscr{M} \rangle\rangle$. One may then invoke the causality condition in (4.12) to argue that $J^+(\mathscr{I}) \setminus \langle\langle \mathscr{M} \rangle\rangle$ and $J^-(\mathscr{I}) \setminus \langle\langle \mathscr{M} \rangle\rangle$ are disjoint, from which it immediately follows that

$$\langle\langle \mathscr{M} \rangle\rangle = J^-(\mathscr{I}) \cap J^+(\mathscr{I}) \,. \tag{5.3}$$

This is analogous to the definition of *the domain of outer communication* in general relativity (see e.g. Carter, 1973), providing further support for the identification $\langle\langle \mathscr{M} \rangle\rangle$ as the 'outside region' of a black/white hole. One may also note that unlike $\langle\langle \mathscr{M} \rangle\rangle$, one may find *only past directed causal curves* from any event in $J^+(\mathscr{I}) \setminus \langle\langle \mathscr{M} \rangle\rangle$ that reaches any arbitrary neighbourhood of $\mathscr{I}$, and likewise, one may find *only future directed causal curves* from any event in $J^-(\mathscr{I}) \setminus \langle\langle \mathscr{M} \rangle\rangle$ that reaches any such neighbourhood of $\mathscr{I}$.

We may now define the *black hole region with respect to* $\mathscr{I}$, to be denoted by $\mathcal{B}(\mathscr{I})$, as the part of the spacetime which is not contained in the past of the boundary at infinity,[1] i.e.

$$\mathcal{B}(\mathscr{I}) \equiv \mathscr{M} \setminus J^-(\mathscr{I}) \ . \tag{5.4}$$

Similarly, we may define the *white hole region with respect to* $\mathscr{I}$, to be denoted by $\mathcal{W}(\mathscr{I})$, as the part of the spacetime which is not contained in the future of the boundary at infinity, i.e.

$$\mathcal{W}(\mathscr{I}) \equiv \mathscr{M} \setminus J^+(\mathscr{I}) \ . \tag{5.5}$$

Finally, the *future and past event horizons* $\mathcal{H}^{\pm}(\mathscr{I})$ of the black and white hole regions with respect to $\mathscr{I}$ will be defined as the boundaries of $J^{\pm}(\mathscr{I})$ in $\mathscr{M}$, respectively, i.e.

$$\mathcal{H}^+(\mathscr{I}) \equiv \partial J^-(\mathscr{I}) \ , \qquad \mathcal{H}^-(\mathscr{I}) \equiv \partial J^+(\mathscr{I}) \ . \tag{5.6}$$

From the general result about the boundaries of past and future sets laid out in (4.16), we may then conclude that both $\mathcal{H}^+(\mathscr{I})$ and $\mathcal{H}^-(\mathscr{I})$ are themselves leaves of the foliation. If $\mathcal{H}^-(\mathscr{I})$ is empty but not $\mathcal{H}^+(\mathscr{I})$, then we will have only a black hole region but no white hole. Likewise, an empty $\mathcal{H}^+(\mathscr{I})$ but non-empty $\mathcal{H}^-(\mathscr{I})$ indicates the presence of a white hole region in the spacetime but no black hole.

The results of equations (5.2) and (5.3), taken alongside the definitions in (5.6), furthermore imply that the only boundaries of $\langle\langle \mathscr{M} \rangle\rangle$ that are in $\mathscr{M}$ are also the event horizons, i.e.

$$\partial\langle\langle \mathscr{M} \rangle\rangle = \mathcal{H}(\mathscr{I}) \equiv \mathcal{H}^+(\mathscr{I}) \cup \mathcal{H}^-(\mathscr{I}) \ . \tag{5.7}$$

1 Though not strictly needed, one might want to add the assumption of *strong asymptotic predictability* here, that would guarantee that the open region $\langle\langle \mathscr{M} \rangle\rangle$ is free of pathologies e.g. 'missing points' and such. This would have to be a suitable adaptation of strong asymptotic predictability as defined in general relativity (see e.g. Wald, 1984, page 299) but with an appropriate notion of development that takes into account the differences in causal structures. We will consider this issue in the next Section, and the concept will be formally introduced in Section 5.2.2.

Being boundaries, $\mathcal{H}^{\pm}(\mathscr{I})$ cannot be contained in $\langle\langle \mathscr{M} \rangle\rangle$ since the latter is an open set by definition; indeed from (4.16) $\mathcal{H}^{+}(\mathscr{I})$ must be in the future of $\langle\langle \mathscr{M} \rangle\rangle$ while $\mathcal{H}^{-}(\mathscr{I})$ must be in the past of $\langle\langle \mathscr{M} \rangle\rangle$. However, since every $\mathfrak{i}_p \in \mathscr{I}$ is simultaneous with some $p \in \Sigma_p \subset \langle\langle \mathscr{M} \rangle\rangle$, causality demands

$$\mathcal{H}^{\pm}(\mathscr{I}) \cap \mathscr{I} = \varnothing \,. \tag{5.8}$$

From the definitions in (5.4) and (5.5), we may further deduce that the black and white hole regions can also be given as

$$\mathcal{B}(\mathscr{I}) = \overline{J^{+}(\mathcal{H}^{+}(\mathscr{I}))}\,, \qquad \mathcal{W}(\mathscr{I}) = \overline{J^{-}(\mathcal{H}^{-}(\mathscr{I}))}\,. \tag{5.9}$$

Both these regions are closed in $\mathscr{M}$, just as in general relativity, since they contain the respective horizons. Clearly, by the causality conditions (4.12), no future-directed causal curve from $\mathcal{B}(\mathscr{I})$ can ever enter $\langle\langle \mathscr{M} \rangle\rangle$, so that $\langle\langle \mathscr{M} \rangle\rangle$ lies outside the future domain of influence of the black hole region, although events in $\langle\langle \mathscr{M} \rangle\rangle$ can causally influence those inside the black hole. Thus, $\mathcal{H}^{+}(\mathscr{I})$ acts as a 'one way causal membrane' separating $\langle\langle \mathscr{M} \rangle\rangle$ from $\mathcal{B}(\mathscr{I})$. In a similar way, no future-directed causal curve from $\langle\langle \mathscr{M} \rangle\rangle$ can ever enter $\mathcal{W}(\mathscr{I})$, hence the white hole region lies beyond the future domain of influence of $\langle\langle \mathscr{M} \rangle\rangle$, although events in $\mathcal{W}(\mathscr{I})$ can causally influence those in $\langle\langle \mathscr{M} \rangle\rangle$. Again, this turns $\mathcal{H}^{-}(\mathscr{I})$ into a 'one way causal membrane' between $\langle\langle \mathscr{M} \rangle\rangle$ and $\mathcal{W}(\mathscr{I})$, though in a sense opposite to $\mathcal{H}^{+}(\mathscr{I})$. Finally, every pair of non-simultaneous events in $\langle\langle \mathscr{M} \rangle\rangle$ are causally connected to each other, such that one event in the pair can always be influenced by the other event through signals sent via causal curves.

Thus far, we have set up a fairly complete framework to address various issues concerning the causal structure of spacetimes with a preferred foliation, where local Lorentz invariance is violated and arbitrarily fast propagation of signals is inherent. Within this framework, we have generalized the notion of a black hole and of a white hole and their corresponding event horizons. An event horizon as defined above traps arbitrarily fast propagations, and therefore provides an appropriate generalization and formalization of the notion of a *universal horizon* which has been introduced in

Blas and Sibiryakov (2011); Barausse et al. (2011). In the rest of this paper, we will refer to event horizons as universal horizons, in order to avoid any possible confusion with Killing and/or null horizons which play the role of event horizons in general relativity. Also, as a final note, we shall point out that the spherically symmetric solutions of Blas and Sibiryakov (2011); Barausse et al. (2011) identified multiple universal horizons nested into each other. Here we will only regards the outermost of such (possible) multiple horizons as *the* universal horizon, since it is the one that truly plays the role of an event horizon, as was made clear in the preceeding discussion.

5.2.2 *Cauchy horizons and event horizons*

In the previous Section, we introduced the notion of event horizons in a manifold with a preferred foliation. In addition, in Section 4.4 we have studied the properties of Cauchy horizons in a spacetime with a preferred foliation; we will then proceed now to prove that universal horizons are necessarily Cauchy horizons – another remarkable feature of manifolds with a preferred foliation.

Towards that end, we will need to make a technical assumption about the spacetimes under consideration: by suitably adopting the corresponding notion from general relativity (see e.g. Wald, 1984, page 299), a foliated spacetime will be called *strongly asymptotically predictable* if $\langle\langle \mathscr{M} \rangle\rangle \subseteq D(\Sigma_p, \mathscr{B})$ for every leaf $\Sigma_p \subset \langle\langle \mathscr{M} \rangle\rangle$, where, as before, $\mathscr{B}$ refers to the relevant part of the collection of all asymptotic boundaries. We then have the following result.

Theorem 3. *In a strongly asymptotically predictable foliated spacetime, the future event horizon $\mathcal{H}^+(\mathscr{I})$ with respect to $\mathscr{I}$ is a future Cauchy horizon of the domain $D^+(\Sigma_p, \mathscr{B})$, i.e.*

$$\mathcal{H}^+(\mathscr{I}) = H^+(\Sigma_p, \mathscr{B}) \ , \quad \forall\, \Sigma_p \subset \langle\langle \mathscr{M} \rangle\rangle \ .$$

Similarly, the past event horizon $\mathcal{H}^-(\mathscr{I})$ *with respect to* $\mathscr{I}$ *is a past Cauchy horizon of the domain* $D^-(\Sigma_p, \mathscr{B})$*, i.e.*

$$\mathcal{H}^-(\mathscr{I}) = H^-(\Sigma_p, \mathscr{B}) \; , \quad \forall\, \Sigma_p \subset \langle\langle \mathscr{M} \rangle\rangle \; .$$

Proof. By the assumption of strong asymptotic predictability as defined above, $\langle\langle \mathscr{M} \rangle\rangle$ is part of the development of every leaf $\Sigma_p \subset \langle\langle \mathscr{M} \rangle\rangle$ and hence cannot contain a Cauchy horizon. Now recall that $\mathcal{H}^\pm(\mathscr{I}) \cap \mathscr{I} = \emptyset$ as already observed in (5.8). Therefore, by Definition 2 of the domain of dependence as well as the arguments presented in the context of case two of Theorem 1, we may conclude that for any leaf $\Sigma_p \subset \langle\langle \mathscr{M} \rangle\rangle$, the leaf $\mathcal{H}^+(\mathscr{I})$ marks the boundary of the future domain of dependence $D^+(\Sigma_p, \mathscr{B})$, and likewise, the leaf $\mathcal{H}^-(\mathscr{I})$ marks the boundary of the past domain of dependence $D^-(\Sigma_p, \mathscr{B})$. □

Theorem 3 demonstrates that universal horizons are always Cauchy horizons; it might be worth emphasising though that the converse is not necessarily true given the much broader definition of Cauchy horizons (e.g. Cauchy horizons may arise in regions not admitting any suitable asymptotic region).

5.3 UNIVERSAL HORIZONS IN STATIONARY SPACETIMES

Up to this point we managed to formalise the notion of a universal horizon in a foliated spacetime $(\mathscr{M}, \Sigma, \mathrm{g})$. Just as in general relativity, this concept is necessarily global; in particular without the knowledge of the entire history of the spacetime, it is impossible to determine whether an hypersurface — or part of one — is a universal horizon. For this reason a more local characterization, if available, is often far more useful in practice. The goal of this Section will be to focus on *stationary spacetimes*, and show that it is indeed possible to characterize universal horizons through their local properties. The underlying (global) 'time translation symmetry' of a stationary spacetime implies that its entire history is always known, and this is the key property that allows for a local characterization. In what follows, we will always assume that every symmetry of the spacetime is satisfied by both the metric and the foliation — and hence the æther — as they are both fundamental elements of any configuration.

In general relativity, an asymptotically flat spacetime is called stationary if it admits a Killing vector whose flow lines are timelike curves 'at least at sufficiently large asymptotic distances' (Carter, 1973). However, such a definition is not satisfactory in our context, since timelike curves have no special meaning in a theory with a preferred foliation and arbitrary speeds of propagation. In fact, whether a certain curve will be timelike or not depends on which one of the speed-c metrics of (4.8) one is willing to use. As such, it would be preferable to have a definition of stationarity which does not make reference to any specific metric.

Let us thus begin by laying down some preliminary terminologies. Suppose $\mathscr{M}$ has an isometry generated by a Killing vector χ^a, whose action will be denoted by $\pi_\chi : \mathbb{R} \times \mathscr{M} \to \mathscr{M}$. The trajectories of this action generates events $\pi_\chi(\tau, p) \in \mathscr{M}$, one for each value of the group parameter $\tau \in \mathbb{R}$ starting with $\pi_\chi(0, p) = p$. As is customary, we will call $\pi_\chi(\tau, p)$ (for all $\tau \in \mathbb{R}$) the orbit of the Killing vector χ^a through the event p. In this work, we will always assume that if $\mathscr{M}$ admits a Killing vector, then its orbits exist everywhere in $\mathscr{M}$. We will similarly denote the action of the isometry on a set of events $\mathscr{Q}$ by $\pi_\chi(\tau, \mathscr{Q})$; e.g. a curve $\lambda(\sigma) \subset \mathscr{M}$ is 'transported' to a curve $\pi_\chi(\tau, \lambda(\sigma)) \subset \mathscr{M}$ under the action of the isometry.

We will call a Killing vector field *causal* in a region of spacetime, if its orbits define causal curves.[2] Since we are working with spacetimes with a trivially foliated flat end, we will assume that the Killing field satisfies the asymptotic conditions

$$(u \cdot \chi) \to -1 \,, \qquad \mathsf{p}_{ab}\chi^a\chi^b \to 0 \,, \tag{5.10}$$

2 Note that for every future directed causal Killing vector χ^a, there exists a past directed causal one given by $-\chi^a$.

near $\mathscr{I}$ (in a suitable sense); this last clause could be made more rigorous by imposing suitable boundary condition on χ^a in an open neighbourhood of i_p for every leaf $\Sigma_p \subset \langle\langle \mathscr{M} \rangle\rangle$ such that (5.10) holds exactly on $\mathscr{I}$. Note that a different asymptotic behaviour needs to be specified for the Killing vector χ^a in order to define stationarity for spacetimes which are not asymptotically flat or have a non-trivial foliation asymptotically; delving deeper into such matters goes beyond our scope though and we will not go into any more detail here.

Let $\mathscr{X} \subseteq \mathscr{M}$ be an open set in the spacetime such that $\mathscr{X} \cup \mathscr{I}$ is the *maximal* connected component of $\mathscr{M} \cup \mathscr{I}$ on which the Killing vector field is causal and future directed; in other words, $(u \cdot \chi) < 0$ everywhere in $\mathscr{X} \cup \mathscr{I}$. We may then propose the following definition of stationarity suitable for the present context:

Definition 3 (Stationary spacetime)**.** A spacetime with a preferred foliation $(\mathscr{M}, \Sigma, g)$ and an open region $\langle\langle \mathscr{M} \rangle\rangle$ admitting a trivially foliated asymptotically flat end will be called stationary if $\mathscr{M}$ admits an isometry generated by a Killing vector χ^a satisfying boundary conditions (5.10) such that

$$\langle\langle \mathscr{M} \rangle\rangle \cap \mathscr{X} \neq \varnothing \,.$$

According to the above definition, in a black/white hole spacetime there exist at least an asymptotic region of the spacetime 'outside' the black/white hole where the Killing vector χ^a is causal. The definition does not rule out the possibility — at least not in an obvious manner — that there could be parts of $\langle\langle \mathscr{M} \rangle\rangle$ where $(u \cdot \chi) \geqslant 0$. Additionally it does not preclude the option that there might be some region of $\mathscr{X}$ continuously connected to $\mathscr{I}$ where the Killing vector is still causal and future directed, but without any overlap with $\langle\langle \mathscr{M} \rangle\rangle$. As will be seen below, a local characterization of a universal horizon will be achieved by analyzing these comments more carefully. Our investigations in this Section has been substantially influenced by the presentation of the analogous results of general relativity in Carter (1973).

As we have already mentioned, static, spherically symmetric and asymptotically flat black hole solutions have been studied in Hořava gravity and Einstein-Æther theory in Barausse et al. (2011); Blas and Sibiryakov (2011), where the notion of the universal horizon was first introduced. Staticity and spherical symmetry make it rather straightforward to identify the universal horizon: in our terminology, it is the outermost location where a leaf of the foliation becomes a constant areal-radius hypersurface; the mere requirement that any signal should travel forward in (preferred) time then implies that such a hypersurface can only be crossed in one direction and no signal from the interior can reach the exterior. A generic feature of all such highly symmetric solutions is that, on the universal horizon one finds that $(u \cdot \chi) = 0$, where χ^a is the Killing vector associated with staticity, that is asymptotically timelike and satisfies $(u \cdot \chi) \to -1$ (recall (5.10)). This strongly suggests $(u \cdot \chi) = 0$ as a condition for the local characterisation of the universal horizon. We will establish below that this is indeed the case, and the condition $(u \cdot \chi) = 0$ (modulo an additional technical assumption) provides a *necessary and sufficient* characterization of a universal horizon in the most general stationary setting. Hence it can be used as a local definition of the universal horizon in stationary systems.

To that end we need some kinematical preliminaries. The acceleration of the æther flow is defined as

$$a_a = u^c \nabla_c u_a = \frac{\boldsymbol{\nabla}_a N}{N} \,, \tag{5.11}$$

where $\boldsymbol{\nabla}_a$ denotes the projected covariant derivative on the foliation leaves. The second equality in (5.11) follows from the hypersurface orthogonality of the æther (see (4.4)). We may then prove the following result which will be of central importance:

Proposition 2. *The hypersurface defined by $(u \cdot \chi) = 0$ and $(a \cdot \chi) \neq 0$ is a leaf of the preferred foliation which cannot be conformally extended to intersect $\mathscr{I}$.*

Proof. The condition that the æther respects stationarity can be expressed as

$$\pounds_\chi u_a = 0 \qquad \Leftrightarrow \qquad \nabla_a(u\cdot\chi) = -(a\cdot\chi)u_a + (u\cdot\chi)a_a\,. \tag{5.12}$$

Since the normal to any $(u\cdot\chi) =$ constant hypersurface is proportional to $\nabla_a(u\cdot\chi)$ by definition, it immediately follows that the hypersurface $(u\cdot\chi) = 0$ is a leaf of the preferred foliation, *provided* $(a\cdot\chi) \neq 0$ everywhere on the same hypersurface. Finally, due to the incompatibility of its defining condition $(u\cdot\chi) = 0$ with the boundary condition (5.10),[3] such a hypersurface cannot be conformally extended to intersect the boundary at infinity $\mathscr{I}$. □

From here onwards, we will always assume that $(a\cdot\chi) \neq 0$ on every event where $(u\cdot\chi) = 0$, as a technical assumption, and will comment on its physical relevance later on. For brevity and convenience, we will use the 'shorthand' Σ_H to denote a leaf defined by the above conditions, namely $(u\cdot\chi) = 0$ and $(a\cdot\chi) \neq 0$. If more than one of such leaves are required to be considered at once, we may distinguish them through additional labels on Σ_H.

As a trivial consequence of the above Proposition 2 and the fact that every leaf in $\langle\langle\mathscr{M}\rangle\rangle$ admits a conformal extension to $\mathscr{I}$ by definition, we immediately have

Corollary 1. Σ_H *can never belong to* $\langle\langle\mathscr{M}\rangle\rangle$ *i.e.*

$$\Sigma_\mathrm{H} \cap \langle\langle\mathscr{M}\rangle\rangle = \emptyset\,.$$

The final theorem which establishes a local characterization of a universal horizon will require some closer investigation of the regions $\langle\langle\mathscr{M}\rangle\rangle$ and $\mathscr{X}$. To that end, the first non-trivial result we need is

Proposition 3.

$$\langle\langle\mathscr{M}\rangle\rangle \setminus \mathscr{X} = \emptyset\,.$$

3 We thank David Mattingly for emphasizing this to us.

Proof. Suppose the contrary and let $p \in \langle\langle \mathscr{M} \rangle\rangle \setminus \mathscr{X}$. By Corollary 1 of Proposition 2, $(u \cdot \chi) \neq 0$ everywhere in $\langle\langle \mathscr{M} \rangle\rangle$. Therefore we must have $(u \cdot \chi) > 0$ on p since $p \notin \mathscr{X}$ by assumption. Now, by definition the region $\langle\langle \mathscr{M} \rangle\rangle$ is the maximal portion of the spacetime $\mathscr{M}$, whose leaves (when conformally extended) intersect the boundary at infinity $\mathscr{I}$. Therefore, *any* event $q \in \Sigma_q \subset \langle\langle \mathscr{M} \rangle\rangle$ can be 'connected' to the point at infinity $\mathrm{i}_q \in \mathscr{I}$ via an acausal curve which is entirely contained in Σ_q. In particular, since $p \in \langle\langle \mathscr{M} \rangle\rangle$ by assumption, there will always be an acausal curve $\lambda(\sigma) \subset \langle\langle \mathscr{M} \rangle\rangle \cup \mathscr{I}$ with $\sigma \in [0,1]$, such that $\lambda(0) = p$, $\lambda(1) \in \mathscr{I}$, and $\lambda(\sigma)$ lies entirely on the (conformal extension of the) leaf Σ_p. Furthermore, the value of $(u \cdot \chi)$ will have to vary in a continuous manner along $\lambda(\sigma)$ starting from some positive number at p (as just argued) to $(u \cdot \chi) = -1$ on $\lambda(1)$ by the boundary condition (5.10). Hence, there has to be an event on $\lambda(\sigma)$ where $(u \cdot \chi) = 0$. But this is a contradiction of Corollary 1 above. Hence $\langle\langle \mathscr{M} \rangle\rangle \setminus \mathscr{X}$ is empty. □

Taken together with Definition 3, an immediate consequence of Proposition 3 is then

$$\langle\langle \mathscr{M} \rangle\rangle \subseteq \mathscr{X} \ .$$

The final result that we need is an upshot of all the preceding results and directly complements Proposition 3. This can be stated as

Proposition 4.

$$\mathscr{X} \setminus \langle\langle \mathscr{M} \rangle\rangle = \emptyset \ .$$

Proof. We have already argued that Proposition 3 implies $\langle\langle \mathscr{M} \rangle\rangle \subseteq \mathscr{X}$. Suppose the stronger result $\langle\langle \mathscr{M} \rangle\rangle \subset \mathscr{X}$ holds, so that $\mathscr{X} \setminus \langle\langle \mathscr{M} \rangle\rangle \neq \emptyset$ contrary to what is claimed above. Then $\langle\langle \mathscr{M} \rangle\rangle$ ends inside $\mathscr{X}$ and *event horizon(s)* $\mathcal{H}(\mathscr{I})$ *must form inside* $\mathscr{X}$ *to mark the end of* $\langle\langle \mathscr{M} \rangle\rangle$ *in* $\mathscr{X}$ (recall, from (5.7), that the event horizons are the only boundaries of $\langle\langle \mathscr{M} \rangle\rangle$ that are actually part of the spacetime). Consequently, the Killing vector χ^a must be causal everywhere on $\mathcal{H}(\mathscr{I}) \subset \mathscr{X}$, in addition to being causal everywhere in

$\langle\langle\mathscr{M}\rangle\rangle$. Note that we are not assuming that both $\mathcal{H}^{\pm}(\mathscr{I})$ are non-empty, but at least one must be in order for $\mathcal{H}(\mathscr{I})$ to be non-empty.

Now, we already noted the existence of acausal curves from *any* event in $\langle\langle\mathscr{M}\rangle\rangle$ which can be 'connected' to the boundary at infinity $\mathscr{I}$. Among the infinitely many such acasual curves, some will also respect the isometry generated by the Killing vector χ^a. For example, due to the asymptotic boundary conditions (5.10), the acceleration a^a of the æther congruence (5.11) tends to 'align' with the canonical radial direction in the 'asymptotic region' (Barausse et al., 2011). However, since the canonical radial vector 'points towards infinity' by definition, at least in a suitably chosen neighborhood of $\mathscr{I}$, integral curves along the acceleration (or along its unit, to be more precise) can 'reach $\mathscr{I}$' as well. Since the acceleration respects the isometry generated by the Killing vector χ^a, at least in a suitably chosen neighborhood of any point at infinity $i_p \in \mathscr{I}$, one may construct an isometry-preserving acausal curve $\lambda(\sigma) \subset \langle\langle\mathscr{M}\rangle\rangle \cup \mathscr{I}$ with $\sigma \in [0,1]$, e.g. along the integral curves of the (unit vector along the) acceleration, such that $\lambda(0) = p \in \Sigma_p \subset \langle\langle\mathscr{M}\rangle\rangle$ and $\lambda(1) \in \mathscr{I}$. Furthermore, due to the isometry-preserving nature of $\lambda(\sigma)$, every member of the family of curves $\pi_\chi(\tau, \lambda(\sigma))$ generated by the group action of the isometry, is acausal in $\langle\langle\mathscr{M}\rangle\rangle$ for every value of the group parameter τ. In particular, since χ^a is causal in $\langle\langle\mathscr{M}\rangle\rangle$ as argued in the preceding paragraph, we may choose the group parameter τ such that the acausal curve $\pi_\chi(\tau, \lambda(\sigma))$ resides in a leaf in the future (past) of Σ_p for a positive (negative) value of τ. Finally, since $\mathscr{I}$ itself is a complete Killing orbit in the conformal extension of the spacetime, we have $\pi_\chi(\tau, \lambda(1)) \in \mathscr{I}$ for all values of the group parameter τ.

By appealing to the assumed strongly asymptotically predictable nature of the region $\langle\langle\mathscr{M}\rangle\rangle$ and employing the causal orbits of the Killing vector χ^a in $\langle\langle\mathscr{M}\rangle\rangle \cup \mathcal{H}(\mathscr{I})$, one may show by a direct adaptation of Proposition 8.3.13 in (Wald, 1984, page 208) that every pair of leaves in $\langle\langle\mathscr{M}\rangle\rangle$ are homeomorphic to each other as well as to the leaf (leaves) $\mathcal{H}(\mathscr{I})$.[4] Therefore, *any* event $p \in \langle\langle \mathscr{M} \rangle\rangle$ can be mapped to some event $q \in \mathcal{H}(\mathscr{I})$ via the map π_χ. Moreover, we may find some event $p \in \langle\langle \mathscr{M} \rangle\rangle$ in some suitably chosen neighborhood of $\mathscr{I}$ through which there exists an isometry-preserving acausal curve $\lambda(\sigma) \in \langle\langle \mathscr{M} \rangle\rangle \cup \mathscr{I}$ as discussed in the preceding paragraph. By transporting $\lambda(\sigma)$ along Killing orbits by the group action in the sense discussed above, one may then generate a curve $\pi_\chi(\tau_0, \lambda(\sigma))$ for some τ_0, such that $\pi_\chi(\tau_0, \lambda(\sigma))$ is acausal, resides on (one of the leaves of) $\mathcal{H}(\mathscr{I})$, and yet $\pi_\chi(\tau_0, \lambda(1)) \in \mathscr{I}$. This is however a direct contradiction of (5.8). Therefore, $\langle\langle \mathscr{M} \rangle\rangle$ cannot be a proper subset of $\mathscr{X}$. □

We are finally in a position to state and prove the central theorem of this section:

Theorem 4 (Local characterization of a universal horizon for a non-extremal black hole)**.** *$(u \cdot \chi) = 0$ and $(a \cdot \chi) \neq 0$ form a set of necessary and sufficient local conditions for a hypersurface to be a universal horizon.*[5]

Proof. By Definition 3, along with the results of Propositions 3 and 4 we have

$$\langle\langle \mathscr{M} \rangle\rangle = \mathscr{X} \ . \tag{5.13}$$

Therefore the boundaries of $\langle\langle \mathscr{M} \rangle\rangle$ and $\mathscr{X}$ in $\mathscr{M}$, $\partial\langle\langle \mathscr{M} \rangle\rangle$ and $\partial\mathscr{X}$ respectively, are identical

$$\partial\langle\langle \mathscr{M} \rangle\rangle = \partial\mathscr{X} \ .$$

As we saw earlier $\partial\langle\langle \mathscr{M} \rangle\rangle = \mathcal{H}(\mathscr{I})$. On the other hand, $\mathscr{X}$ is the maximal open set in $\mathscr{M}$ connected to $\mathscr{I}$ where the Killing vector is causal and future directed; hence we must have $(u \cdot \chi) = 0$ on $\partial\mathscr{X}$. In other words, $(u \cdot \chi) = 0$ is an appropriate local characterization for event horizons under the assumptions of Proposition 2. □

4 Actually, since the orbits of the Killing vector χ^a are smooth curves by assumption, we have a diffeomorphism between every pair of leaves in $\langle\langle \mathscr{M} \rangle\rangle$, which is stronger statement. However, this observation will not be needed in the main proof.

5 Notice that we defined this explicitely for a non-extremal black hole. We will comment in the following about the reason for this requirement.

It also seems reasonable that a very similar local definition of the universal horizon should exist for other kinds of asymptotic behaviour of the spacetime, for example in solutions with maximally symmetric asymptotics (Bhattacharyya and Mattingly, 2014) or Lifshitz asymptotics (Griffin et al., 2013; Janiszewski, 2015; Basu et al., 2016). On the other hand, note that the æther defines a geodesic if the acceleration (5.11) vanishes globally; this is true, for instance, in the *projectable* version of Hořava gravity (see e.g. Hořava (2009b); Sotiriou et al. (2009a,b)). For such solutions $(u \cdot \chi) = -1$ globally (for the asymptotic boundary conditions of (5.10) assumed here) and, hence, $(u \cdot \chi)$ cannot vanish anywhere. By Theorem 4, such spacetimes cannot admit universal horizons.

Theorem 4 guarantees that a universal horizon will be stationary (i.e. contains the Killing vector χ^a as one of the generators of the horizon), much like event horizons in stationary spacetimes in general relativistic theories are Killing horizons. In fact, it is instructive to compare the local condition $(u \cdot \chi) = 0$ with the condition $\chi^2 = 0$ which identifies Killing horizons in general relativity. Indeed, one can have multiple leaves of the foliation on which the condition $(u \cdot \chi) = 0$ holds (see e.g. the solutions in Barausse et al., 2011), in the same way as in general relativity where one can have multiple Killing horizons (e.g. in Reissner-Nordström or Kerr solutions). In both cases, the outermost of these acts as the event horizon.

The seemingly technical assumption $(a \cdot \chi) \neq 0$, that goes together with the local characterization of universal horizons in Theorem 4, has an important physical significance. It has been argued (Cropp et al., 2014; Berglund et al., 2013) that $(a \cdot \chi)$ plays the role of the surface gravity associated with a universal horizon. We will now demonstrate that a non-zero $(a \cdot \chi)$ is always constant on a universal horizon where $(u \cdot \chi) = 0$ by Theorem 4. This establishes a further strong parallel with the so called *zeroth law of black hole* *thermodynamics* (Bardeen et al., 1973).[6] The acceleration is built out of the æther and the metric and, as such, it has vanishing Lie derivative along χ^a. In particular, the condition analogous to (5.12) is

$$\pounds_\chi a_a = 0 \qquad \Leftrightarrow \qquad \boldsymbol{\nabla}_a (a \cdot \chi) = (u \cdot \chi) \pounds_u a_a \,. \tag{5.14}$$

Clearly, if $(a \cdot \chi) \neq 0$, then it is constant on a leaf defined by $(u \cdot \chi) = 0$ and hence on a universal horizon according to Theorem 4. Invoking the parallel with surface gravity, a non-vanishing $(a \cdot \chi)$ thus characterizes a *non-degenerate universal horizon* which is analogous to a non-degenerate Killing horizon.

At this point, it is worth mentioning the issue with extremal black holes. In GR extremal black holes are solutions with vanishing surface gravity. While not achiavable in practice, such black holes are well defined and necessary in discussing black hole theormodynamics. On the other hand, if $(a \cdot \chi)$ indeed represents the surface gravity in our case, it would seem like Theorem 4 actually prevents the existence of extremal black holes since it is explicitely required that $(a \cdot \chi) \neq 0$. The origin of the conundrum can be found in Proposition 2. Here we required the condition $(a \cdot \chi) \neq 0$ to hold in order to prove that the surface where $(u \cdot \chi) = 0$ is indeed a leaf of the foliation. It may be possible to show that the condition on $(a \cdot \chi)$ can be done away with and therefore including extremal black holes in the definition of Theorem 4. This is still unclear though, and therefore we chose to restrict the definition to non-extremal horizons. More work is thus required in order to understand if extremal horizons can be included in such definition, or if they should be defined in a different way.

Now that we have some insight into the meaning of the $(a \cdot \chi) \neq 0$ condition, it is worth exploring a bit further the implications of the $(u \cdot \chi) = 0$

6 See Berglund et al. (2012); Bhattacharyya and Mattingly (2014) for a derivation of the laws of black hole mechanics for spherically symmetric Einstein-Æther/Hořava gravity solutions.

condition itself. By projecting the identity in (5.12) on a leaf of the foliation, one obtains

$$\boldsymbol{\nabla}_a(u\cdot\chi) = a_a(u\cdot\chi)\ . \tag{5.15}$$

Combining (5.11) and (5.15), one obtains then

$$N = f(T)(u\cdot\chi)\ , \tag{5.16}$$

where $f(T)$ is some undetermined function of the preferred time function T and N is the lapse of the foliation. Since $(u\cdot\chi)=0$ at the universal horizon, either N has to vanish as well, or $f(T)$ has to diverge there. In a theory where the foliation leaves are uniquely labelled and time-reparametrizations are not allowed, $f(T)$ is fixed by asymptotics. In particular, the asymptotic conditions $(u\cdot\chi) = -1$ and $N = 1$ imply $N = -(u\cdot\chi)$ for any leaf that reaches $\mathscr{I}$. Then, by continuity, the lapse would have to vanish on the universal horizon rendering the foliation singular. Hence, regular universal horizons cannot exist in theories that do not enjoy reparametrization invariance. When time reparametrizations are allowed instead, a divergent $f(T)$ is not worrisome. In the time parametrization that satisfies the asymptotic conditions, which is in the time of a preferred observer at infinity, the lapse would have to vanish. But a suitable time reparametrization would lead to a non-zero lapse. Recall that both N (see (4.7)) and $f(T)$ will transformation under (4.3), leaving $(u\cdot\chi)$ unaffected.

There is one more issue that needs addressing before our local characterisation can be considered meaningful. Namely, it should not depend on which causal Killing vector one chooses to use, else there would be an ambiguity regarding this choice. Let ξ^a denote a causal Killing vector which is not proportional to χ^a. Being a Killing vector, ξ^a satisfies an exact analogue of (5.12). Consequently, analogous to (5.16), we have $N = g(T)(u\cdot\xi)$ for some function $g(T)$ possibly different from $f(T)$. However, since by assumption the foliation is ordered but not labeled, and both $(u\cdot\chi)$ and $(u\cdot\xi)$ are invariant under the time-reparametrizations in (4.3), the above must imply $(u \cdot \xi) = C_0(u \cdot \chi)$ for some non-zero constant C_0. In other words, $(u \cdot \xi)$ must indeed vanish whenever $(u \cdot \chi)$ vanishes.

Based on the above observations, one can derive some very useful properties of Killing vectors in foliated spacetimes. We may begin by noting that the linear combination $\xi^a - C_0\chi^a$, which itself is a Killing vector, must be orthogonal to the æther, and hence acausal everywhere. Therefore, for every causal Killing vector ξ^a linearly independent of χ^a, there exists an acausal Killing vector ϕ^a such that

$$\xi^a = C_0\chi^a + \phi^a \,, \qquad (u \cdot \phi) = 0 \,, \qquad C_0 = \text{constant} \,. \tag{5.17}$$

By the above relation, one may always subtract the χ^a-component of any such causal Killing vector, thereby reducing it to an acausal Killing vector. Consequently, it suffices to regard χ^a as the only causal Killing vector in a stationary spacetime with a preferred foliation, and the existence of any other linearly independent causal Killing vector signifies the existence of an additional symmetry generated by an acausal Killing vector. Furthermore, since the æther also satisfies the symmetry generated by ϕ^a, manipulations analogous to those leading to (5.12) yields

$$\pounds_\phi u_a = 0 \qquad \Leftrightarrow \qquad (a \cdot \phi) = 0 \,. \tag{5.18}$$

The content of conditions (5.17) and (5.18) can be summarized as follows: *every acausal Killing vector ϕ^a is also orthogonal to the acceleration of the æther and therefore can only span the n-dimensional subspace orthogonal to both the æther and its acceleration.*

Given two Killing vectors χ^a and ϕ^a, with the former being causal and the latter acausal without any loss in generality, a standard method to generate (potentially new) Killing vectors is by considering their commutator; this is because, $\psi^a = \pounds_\chi \phi^a$ if non-zero is a Killing vector. However, since ϕ^a is acausal and the æther respects stationarity, we may immediately conclude that ψ^a is acausal as well

$$\pounds_\chi (u \cdot \phi) = (u \cdot \psi) = 0 \,, \qquad (a \cdot \psi) = 0 \,, \tag{5.19}$$

with the second condition being a direct consequence of (5.18). Note that our conclusions remain trivially valid if χ^a were to commute with ϕ^a. This particular observation will be useful in the next section. Finally, given a pair of acausal Killing vectors ϕ^a and ψ^a whose symmetries are respected by the æther, their commutator $\theta^a = \pounds_\phi \psi^a$ is also acausal, because

$$\pounds_\phi (u \cdot \psi) = (u \cdot \theta) = 0\,, \qquad (a \cdot \theta) = 0\,, \tag{5.20}$$

the second condition, once again, being a consequence of (5.18). Summing up the observations in (5.17)-(5.20), we may also note that *the actions of acausal Killing vectors are always confined within the leaves of the foliation*. These observations can be utilized to study the algebra of symmetries compatible with foliated spacetimes $(\mathscr{M}, \Sigma, \mathrm{g})$. We leave this for future investigations.

5.4 EXISTENCE OF KILLING HORIZONS IN STATIONARY AXISYMMETRIC SPACETIMES WITH A UNIVERSAL HORIZON

So far in this work, we have studied the causal structure of a spacetime $\mathscr{M}$ with a metric g_{ab} and a preferred foliation structure Σ. In particular, we focused on those issues of causality which are strongly tied to the preferred foliation, and essentially argued that the spacetime metric g_{ab} is of little relevance when it comes to the global causal structure of $\mathscr{M}$. In fact, this last observation applies equally well to any of the speed-c metrics $\mathrm{g}^{(c)}_{ab}$ as defined in (4.8). A specific example of the irrelevance of the metrics is provided by the local characterization of a universal horizon given in Theorem 4, which only involves the inner product between the æther one-form u_a and the Killing vector generating stationarity χ^a, without making any reference to any metric. One may compare the above situation with general relativity where a stationary event horizon is a Killing horizon (Carter, 1969, 1973) and the latter is an intrinsically metric dependent notion. In this section we will explore the existence and the role of Killing horizons within our framework.

We will restrict our attention to spacetimes that are not only stationary but also axisymmetric, as this significantly simplifies calculations. In general relativity the celebrated Hawking rigidity theorem (Hawking, 1972, 1973) establishes that under certain reasonable assumptions, stationary black holes in general relativity must be axisymmetric. However, the assumptions that underlie Hawking's theorem include the Weak Energy Condition for matter fields and the existence of a bifurcation surface. These assumptions are not necessarily satisfied outside the framework of general relativity and it is not clear whether Hawking's theorem can be generalized. Hence, in our framework, axisymmetry will have to be an extra assumption. The fact that quiescent, rotating black holes are expected to be axisymmetric provides the necessary motivation for making such an assumption.

It is worth mentioning once again that, in the special case of the two-derivative (low-energy) version of Hořava gravity, slowly-rotating axisymmetric solutions have been found and they naturally extend the much studied spherically symmetric solution space (Barausse and Sotiriou, 2012, 2013a,b). Additionally, certain stationary axisymmetric solutions of Hořava gravity in $(1+2)$-dimensions are already known (Sotiriou et al., 2014) so the following analysis should also pave the way towards a comparison of these solutions with their $(1+3)$ dimensional counterparts.

We will then begin by establishing the general properties of a stationary, axisymmetric, foliated spacetime $(\mathscr{M}, \Sigma, \mathrm{g})$. Let us denote the stationarity generating Killing vector by χ^a as before, and let φ^a be the Killing vector generating axisymmetry. As explained previously, see conditions (5.17) and (5.18), φ^a can be taken to be acausal without any loss of generality, which means

$$(u \cdot \varphi) = (a \cdot \varphi) = 0 \,. \tag{5.21}$$

These conditions thus also naturally avoid any violation of causality (recall Proposition 1), since φ^a has closed orbits as a generator of axisymmetry.

Furthermore, given that we wish to consider asymptotically flat spacetimes, the Killing vectors χ^a and φ^a commute (Carter, 1970, 1973), i.e.

$$\pounds_\chi \varphi^a = \pounds_\varphi \chi^a = 0 \,. \tag{5.22}$$

Now, consider a foliated spacetime arising as a solution of a theory with a preferred foliation (e.g. Hořava gravity). In accordance with our previous discussions, assume that the spacetime has an open region $\langle\langle \mathscr{M} \rangle\rangle$ with a trivially foliated flat end, satisfies stationarity and axisymmetry, and admits a future universal horizon. By Theorem 4, such a future universal horizon is characterized by a leaf on which $(u \cdot \chi) = 0$, while φ^a plays no role in this definition. Assume, furthermore, that some matter field propagates in such a background which couples minimally to a speed-c metric $\mathsf{g}^{(c)}_{ab}$ for some fixed c (we could consider as an example $c = 1$ for concreteness). Such a matter field will then only 'feel' an effectively quasi-relativistic causal structure of the spacetime dictated by the propagation cones of $\mathsf{g}^{(c)}_{ab}$, instead of the more fundamental causal structure dictated by the preferred foliation, due to the second-order equations of motion/dispersion relations arising from the matter field's minimal coupling with the speed-c metric. Therefore, quasi-relativistic features of the spacetime geometry governed by $\mathsf{g}^{(c)}_{ab}$ are expected to play a significant role in the propagation of such matter fields; e.g. *null (event) horizons* should define the regions of the spacetime which can be causally accessed and/or influenced by such matter fields.

This last fact is rendered even more important by the existence of the Killing vector χ^a with all its properties. More specifically, with respect to *every* speed-c metric, χ^a is timelike asymptotically due to the boundary condition (5.10) while it is spacelike on the universal horizon due to being orthogonal to the timelike æther there. Therefore, in particular, somewhere in the bulk of the spacetime χ^a must turn null with respect to the speed-c metric $\mathsf{g}^{(c)}_{ab}$ coupling minimally to the matter field; in addition the surface $\mathsf{g}^{(c)}_{ab}\chi^a\chi^b = 0$ must remain *outside* the universal horizon by the assumed smoothness of the background spacetime. In the special case of spherically symmetric solutions (Barausse et al., 2011; Bhattacharyya and Mattingly, 2014), such a surface is also a null hypersurface, making it a Killing horizon of $g^{(c)}_{ab}$. Consequently in a static, spherically symmetric and asymptotically flat spacetime with a universal horizon, matter fields coupling minimally to some speed-c metric will see a Killing horizon outside the universal horizon, with the former already acting as an effective causal barrier for such fields.

More generally however, a surface on which χ^a turns null with respect to $g^{(c)}_{ab}$ is not necessarily a Killing horizon of $g^{(c)}_{ab}$, but just an ergosurface. This raises the following questions:

1. Does the existence of a universal horizon in a stationary, axisymmetric and asymptotically flat spacetime (with a trivially foliated flat end) imply the existence of a Killing horizon, at least of some speed-c metric?

2. If such a Killing horizon exists, does it necessarily lie outside the universal horizon?

It should be stressed that answering these two questions is of important physical significance. If the low-energy modes of a given excitation see no Killing horizon before reaching the universal horizon, then one could have a fairly significant departure from relativistic physics at low energies since the structure of black hole spacetime would be changed with respect to the usual GR solutions; in particular to give an example the position of the horizon would not be compatible with what we would expect. Additionally, the thermodynamics of such black holes would change drastically.

As a preparation towards tackling the above questions, let us introduce the vector V^a defined by the following linear combination of χ^a and φ^a,

$$V^a = \chi^a + W\varphi^a \, , \tag{5.23}$$

where

$$W = -(\chi \cdot \varphi)(\varphi \cdot \varphi)^{-1} \, . \tag{5.24}$$

Due to the acausal nature of φ^a [see (5.21)], the inner products $(\chi \cdot \varphi)$ and $(\varphi \cdot \varphi)$ are the same with respect to any speed-c metric (4.8), hence the same applies to W. Note that V^a is not a Killing vector in general, since $W \neq$ constant. V^a is also orthogonal to φ^a

$$(V \cdot \varphi) = 0 , \tag{5.25}$$

by construction; this also holds with respect to any speed-c metric. Additionally, we have the following relations as straightforward consequences of (5.21) and (5.23)

$$(u \cdot V) = (u \cdot \chi) , \qquad (a \cdot V) = (a \cdot \chi) . \tag{5.26}$$

Note that the above relations also do not require any metric since both the æther in (4.4) and the acceleration in (5.11) are naturally defined as one-forms, while V^a in (5.23) and χ^a are naturally given as vectors. If we furthermore define the projections of V^a and χ^a orthogonal to the æther, which will be purely spatial by construction, as follows

$$\boldsymbol{V}^a = \mathsf{p}^a{}_b V^b , \qquad \boldsymbol{\chi}^a = \mathsf{p}^a{}_b \chi^b , \tag{5.27}$$

then the analogous projection of (5.23) becomes

$$\boldsymbol{V}^a = \boldsymbol{\chi}^a + W \varphi^a . \tag{5.28}$$

Exploiting the orthogonality of V^a and φ^a, we then have the following identity for the norms

$$\boldsymbol{\chi}^2 = \boldsymbol{V}^2 + W^2 \varphi^2 . \tag{5.29}$$

Every norm in the above equation is positive semi-definite, since all the vectors are purely spatial. For essentially the same reason, the norms are unchanged when computed with respect to any speed-c metric, and therefore so is the entire relation in (5.29).

Introducing the vector V^a in (5.23) is particularly helpful because one can then use Theorem 4.2 of Carter (1973) as well as its Corollary (the latter may be referred to as 'Carter's rigidity theorem') in order to establish the nature of the hypersurface where $\mathsf{g}_{ab}V^aV^b = 0$. In particular, if the commuting Killing vectors χ^a and φ^a [see (5.22)] satisfy the *circularity conditions*[7]

$$\chi_{[a}\varphi_b\nabla_c\varphi_{d]} = 0\,, \qquad \varphi_{[a}\chi_b\nabla_c\chi_{d]} = 0\,, \tag{5.30}$$

then the hypersurface $\mathsf{g}_{ab}V^aV^b = 0$ is null, and V^a is a Killing vector on said hypersurface — equivalently W = constant on the $\mathsf{g}_{ab}V^aV^b = 0$ hypersurface — so that the hypersurface is a Killing horizon. A very important and relevant aspect of the above result is its purely geometrical nature, appealing neither to any equations of motion, nor to any specific energy conditions. Additionally, even though the above result is specifically stated with respect to the metric g_{ab}, it can be generalized for any speed-c metric. In particular, the analogue of (5.30) is obtained by replacing the Killing one-forms χ_a and φ_a in (5.30) with those obtained by 'lowering the indices' of the corresponding Killing vectors with the appropriate speed-c metric (e.g. see (5.48) below). Similarly, for scalar relations, norms and inner-products needs to be computed with respect to the same speed-c metric, e.g. replace $\mathsf{g}_{ab}V^aV^b$ with $\mathsf{g}^{(c)}_{ab}V^aV^b$ etc.

Suppose now that we are given a non-trivial stationary and axisymmetric spacetime, with an open region having a trivially foliated flat end and admitting a future universal horizon, where the Killing vectors satisfy the circularity conditions in (5.30). The asymptotic boundary conditions of (5.10), taken together with the identities (5.26) and (5.29), then imply $(u \cdot V) \to -1$ and $\mathsf{p}_{ab}V^aV^b \to 0$ asymptotically, while the existence of a future universal horizon means $(u \cdot V) = 0$ there by (5.26) and Theorem 4. Therefore, V^a is timelike asymptotically but turns spacelike on the universal horizon, just like χ^a. By the smoothness of the background, V^a must turn null somewhere and the surface $\mathsf{g}_{ab}V^aV^b = 0$ must exist outside the universal horizon. Since the Killing vectors satisfy the circularity conditions by assumption, the theorems in Carter (1973) discussed above imply that the surface $\mathsf{g}_{ab}V^aV^b = 0$

7 The Theorem also requires that the open region $\langle\langle \mathscr{M} \rangle\rangle$ with a trivially foliated flat end be *simply connected*. We assume this to hold in what follows on physical grounds.

is a Killing horizon with respect to the metric g_{ab}. Once again, these statements have straightforward generalizations to all speed-c metrics.

The above show that a Killing horizon of any speed-c metric will lie outside the universal horizon, provided one does form. More importantly, they also show that a Killing horizon of a certain speed-c metric will always form in a spacetime with suitable properties provided that the Killing vectors in that spacetime satisfy the circularity relations (5.30) with respect to the same speed-c metric. Whether this will be the case or not will depend on the dynamics of the gravity theory in question. This can be seen rather straightforwardly by suitably re-expressing the circularity relations. We will explicitly work with the metric g_{ab} for most part, but our results will hold directly with any speed-c metric.

We may begin by noting that by the definitions of the *twists* of the Killing vectors (Wald, 1984)

$$\varpi^a = \varepsilon^{abcd}\varphi_b(\nabla_c\varphi_d)\,, \qquad \omega^a = \varepsilon^{abcd}\chi_b(\nabla_c\chi_d)\,, \tag{5.31}$$

the circularity conditions (5.30) can be equivalently expressed as

$$(\varpi\cdot\chi) = 0\,, \qquad (\omega\cdot\varphi) = 0\,. \tag{5.32}$$

Now, a standard identity in differential geometry involving a pair of commuting Killing vectors and their twists, valid irrespective of the circularity conditions (5.32), states that (Wald, 1984, Theorem 7.1.1)

$$\begin{aligned} \varepsilon_{abcd}\chi^b\varphi^c\mathcal{R}^d{}_e\varphi^e &= \nabla_a\left[-\frac{1}{2}(\varpi\cdot\chi)\right]\,, \\ \varepsilon_{abcd}\varphi^b\chi^c\mathcal{R}^d{}_e\chi^e &= \nabla_a\left[-\frac{1}{2}(\omega\cdot\varphi)\right]\,, \end{aligned} \tag{5.33}$$

where $\mathcal{R}_{ab}$ is the Ricci tensor of the metric we are considering. Therefore, any stationary and axisymmetric spacetime satisfying

$$\varepsilon_{abcd}\chi^b\varphi^c\mathcal{R}^d{}_e\varphi^e = 0\,, \qquad \varepsilon_{abcd}\varphi^b\chi^c\mathcal{R}^d{}_e\chi^e = 0\,, \tag{5.34}$$

guarantees the circularity conditions (5.32) – and hence (5.30) – globally, since (5.32) holds at least on the rotation axis where φ^a vanishes (Wald, 1984; Carter, 1973, 1970). The conditions (5.30), (5.32) and (5.34) are thus all physically equivalent. On the other hand, the conditions (5.34) are directly related to the dynamics of the underlying theory by virtue of the generalized Einstein's equations.

Stationary and axisymmetric *vacuum* solutions in general relativity satisfy the conditions (5.34) trivially, and a similar conclusion can be drawn for stationary and axisymmetric *electro-vacuum* solutions in general relativity with a little more effort (Wald, 1984; Carter, 1973). More generally however, in a theory with a different matter content, the vectors $\varepsilon_{abcd}\chi^b\varphi^c\mathcal{T}^d_{\ e}\,\varphi^e$ and $\varepsilon_{abcd}\varphi^b\chi^c\mathcal{T}^d_{\ e}\,\chi^e$ built out of the matter stress tensor $\mathcal{T}_{ab}$ may not vanish identically everywhere in a stationary axisymmetric spacetime. Consequently, the conditions (5.34) may fail to hold by Einstein's equations, and such geometries will not satisfy the circularity conditions (5.30) globally.

It is worth stressing at this point that, if the circularity conditions fail to hold, then there is no coordinate system in which a stationary, axisymmetric metric will take the usual (Papapetrou) form, with $g_{t\phi}$ being the only non-vanishing off-diagonal component (Wald, 1984; Papapetrou, 1953). Hence one would expect this condition to hold for black holes whose spacetime structure is sufficiently close to those of general relativity. Said otherwise, one can expect significant deviations from general relativity in theories where the circularity conditions do not hold.

5.4.1 *Circularity conditions: the case of Hořava gravity*

In order to go any further, we will need to choose a particular theory of gravity. Thus, we will concentrate from here onwards on the two-derivative version of Hořava gravity in order to demonstrate that the circularity conditions are in general not trivially satisfied.

As a first thing, before going on to the calculations we want to perform, we need to set up the theory. Since we are discussing black holes, the Ricci curvature of spacetime —at least far away from the very centre — is not too extreme, and therefore we can safely assume to be in a low-energy regime. For this reason we have no use in considering the full blown version of Hořava gravity. In fact doing so would make our life much more complicated, due to the high number of terms we would need to consider, and at the same time we would be using a lot of terms which we can safely discard since they are negligible. In addition, thanks to the presence of the instantaneous mode discussed in Blas and Sibiryakov (2011) — which we will also describe more precisely in the next Chapter — such two derivative truncation still has the same causal structure as the full version of Hořava gravity, and therefore its black hole solutions will anyway retain all the characteristics we discussed before.

For this reason we will, in this last part of the present Chapter, consider only the low-energy version of Hořava gravity truncated at second order derivatives. Additionally for convenience we will use the covariant version described in Section 2.2.3 of Chapter 2. We will not discuss here how this version of the theory is obtained, the reader is invited to refer to Chapter 2 for the details on such procedure, but it might be useful to at least remind ourselves of the for the action of this version of the theory assumes.

The action for the two-derivative truncation of Hořava gravity can be expressed in a covariant manner — modulo boundary terms — as

$$S_{\mathrm{HL}} = \frac{1}{16\pi G_{\mathrm{HL}}} \int \mathrm{d}^4x \sqrt{-\mathrm{g}}\, [\mathcal{R} + \mathcal{L}_{\mathrm{HL}}] \ , \tag{5.35}$$

where G_{HL} is a dimensionful normalization constant with the same dimensions of the Newton's constant, the scalar curvature $\mathcal{R}$ represents the standard Einstein-Hilbert term and $\mathcal{L}_{\mathrm{HL}}$ is the lagrangian for the æther defined by (see (2.49) in Section 2.2.3 of Chapter 2)

$$\mathcal{L}_{\mathrm{HL}} = -c_{13} K_{ab} K^{ab} - c_2 K^2 + c_{14} a^2 \ , \tag{5.36}$$

Now as a side comment before proceeding any further, remember that since we are considering Hořava gravity, the æther vector u^a has to be hypersurface orthogonal; this is ensured by the fact that the fundamental variable, other than the metric g_{ab}, is actually the khronon T, hidden in the æther vector through the definition $u_a = -N\nabla_a T$.

Extremizing the action (5.35) with respect to variations of the metric g_{ab} yields the generalised Einstein equations

$$\mathcal{R}_{ab} = \mathcal{T}_{ab} - \frac{\mathcal{T}}{2} g_{ab} \,, \tag{5.37}$$

where $\mathcal{T}_{ab}$ is the *khronon's stress tensor* obtained by varying the lagrangian (5.36) with respect to the metric, and $\mathcal{T}$ is its trace. Variation of the action with respect to the scalar field T gives rise to its equation of motion. However, we will not need to use of this equation here. In fact, due to the diffeomorphism invariance of the action (5.35), the contracted Bianchi identity implies the equations of motion for the khronon T when the Einstein equations (5.37) are satisfied (Barausse et al., 2011; Jacobson, 2011). As a result, the Einstein equations (5.37) are sufficient to completely determine the spacetime together with the preferred foliation.

Let us now return to the discussion about the circularity condition for Killing vectors is stationary, axisymmetric configuration. One may compute the quantities $\varepsilon_{abcd}\chi^b\varphi^c\mathcal{T}^d_{\ e}\,\varphi^e$ and $\varepsilon_{abcd}\varphi^b\chi^c\mathcal{T}^d_{\ e}\,\chi^e$ using the expression for the khronon's stress tensor. If both of these quantities vanish, the Einstein's equations (5.37) will imply the circularity conditions (5.30) or (5.32) via the equivalent identity (5.34). Conversely, the circularity conditions will fail to hold if either or both of $\varepsilon_{abcd}\chi^b\varphi^c\mathcal{T}^d_{\ e}\,\varphi^e$ and $\varepsilon_{abcd}\varphi^b\chi^c\mathcal{T}^d_{\ e}\,\chi^e$ are non-zero. Our remaining task is then to evaluate these expressions, and this constitutes the 'strategy' to examine the validity of the circularity conditions for the Killing vectors in the present context.

It is convenient to decompose the khronon's stress tensor $\mathcal{T}_{ab}$ in the preferred frame as follows

$$\mathcal{T}_{ab} = \mathcal{T}_{uu}u_a u_b - (u_a\mathbf{\Pi}_b + u_b\mathbf{\Pi}_a) + \boldsymbol{T}_{ab} \,, \tag{5.38}$$

where the 'purely spatial' components (i.e. those which are orthogonal to the æther) are defined as

$$\mathbf{\Pi}_a = \mathsf{p}_a{}^c u^d \mathcal{T}_{cd} \,, \qquad \boldsymbol{T}_{ab} = \mathsf{p}_a{}^c \mathsf{p}_b{}^d \mathcal{T}_{cd} \,. \tag{5.39}$$

From the variation of the lagrangian (5.36) with respect to the metric, the individual 'components' of the decomposition in (5.38) can be computed. To begin with, the 'energy density' $\mathcal{T}_{uu}$ with respect to the preferred frame is given by

$$\mathcal{T}_{uu} = \frac{\mathcal{L}_{\text{HL}}}{2} + c_{14}(\boldsymbol{\nabla} \cdot a) \; . \tag{5.40}$$

Next, by projecting out the Einstein's equations (5.37) analogous to the definition (5.39) of $\boldsymbol{\Pi}_a$, one has[8]

$$(1 - c_{13})\boldsymbol{\nabla}_c K^c{}_a = (1 + c_2)\boldsymbol{\nabla}_a K \; . \tag{5.41}$$

Making use of the above, the 'cross components' $\boldsymbol{\Pi}_a$ turn out to take the following form

$$\boldsymbol{\Pi}_a = \frac{c_{123}}{1 - c_{13}} \boldsymbol{\nabla}_a K \; . \tag{5.42}$$

Since $\boldsymbol{\Pi}_a$ is a purely spatial gradient and respects the Killing symmetry generated by the purely spatial Killing vector φ^a, we have

$$\pounds_\varphi K = 0 \qquad \Leftrightarrow \qquad \boldsymbol{\Pi} \cdot \varphi = 0 \; , \tag{5.43}$$

Finally, the completely spatial part $\boldsymbol{T}_{ab}$ of the khronon's stress tensor is given by

$$\boldsymbol{T}_{ab} = \left(c_2 \nabla_c [K u^c] + \frac{\mathcal{L}_{\text{HL}}}{2} \right) \mathsf{p}_{ab} - c_{14} a_a a_b + c_{13} \left(K K_{ab} + \pounds_u K_{ab} - 2 K_a{}^c K_{bc} \right) \; . \tag{5.44}$$

For future convenience, let us also decompose the twist ϖ_a (5.31) of the Killing vector φ^a in its components along and perpendicular to the æther hypersurfaces as follows

$$\varpi^a = \varpi^{(3)} u^a + \boldsymbol{\varpi}^a \; , \tag{5.45}$$

where the scalar $\varpi^{(3)}$ and the purely spatial vector $\boldsymbol{\varpi}_a$ are defined as

$$\varpi^{(3)} = \varepsilon^{abcd} \varphi_a (\boldsymbol{\nabla}_b \varphi_c) u_d \; , \qquad \boldsymbol{\varpi}^a = 2 \varepsilon^{abcd} \varphi_b K_{ce} \varphi^e u_d \; . \tag{5.46}$$

8 It can be shown (Donnelly and Jacobson, 2011) that (5.41) is also the momentum constraint equation in a Hamiltonian formulation of Hořava gravity adapted to the preferred foliation.

In particular, $\varpi^{(3)}$ is the twist of φ^a on each leaf of the foliation. In terms of the above, a direct computation yields

$$\varepsilon_{abcd}\chi^b\varphi^c\mathcal{T}^d_{\ e}\,\varphi^e = \nabla_a\left[-\frac{c_{13}}{2}(\boldsymbol{\varpi}\cdot\chi)\right] . \tag{5.47}$$

It should be clear that the right hand side of this equation does not vanish for $c_{13} \neq 0$ without imposing the further condition $\boldsymbol{\varpi}\cdot\chi$ =constant. Hence, the circularity condition for φ^a is not trivially satisfied. Since φ^a vanishes on the axis of rotation, $\boldsymbol{\varpi}\cdot\chi$ has to vanish there are well, and the requirement for the circularity condition to hold reduces to $\boldsymbol{\varpi}\cdot\chi = 0$. To get a bit more insight of what this further condition implies for the foliation, one can combine (5.33) and (5.47), to obtain the relation

$$\varpi^{(3)}(u\cdot\chi) + (1-c_{13})(\boldsymbol{\varpi}\cdot\chi) = 0\,, \tag{5.48}$$

where we have again used the fact that φ^a vanishes on the rotation axis. Using this relation, it becomes clear that $\boldsymbol{\varpi}\cdot\chi = 0$ implies $\varpi^{(3)} = 0$, and hence, φ would have to always reside in the foliation leaves and actually be normal to a set of surfaces that foliate the leaf.

Note that the discussion above has been conducted in terms of a specific speed-c metric. The results, however, qualitatively apply to all speed-c metrics, with one exception: the *spin-*2 *metric*, $\mathrm{g}_{ab}^{(c_{\rm spin\,2})}$. Low-energy spin-2 perturbation in Hořava gravity propagate along null surface of this metric. Indeed, one can use the transformation introduced in Foster (2005) in order to set $c_{13} = 0$, but this would be equivalent to working with the spin-2 metric. In other words, the circularity condition for the Killing vector φ^a *does* hold globally with respect to the spin-2 metric, which is rather remarkable. In fact, one may directly confirm that the condition (5.48) is the analogue of the circularity condition $(\varpi\cdot\chi) = 0$ in (5.32), but with respect to this metric only.

However, for the hypersurface $\mathrm{g}_{ab}^{(c_{\rm spin\,2})}V^aV^b = 0$ to turn into a Killing horizon one further needs to verify whether the remaining circularity condition for χ^a, namely $(\omega\cdot\varphi) = 0$ as in (5.32), holds with respect to $\mathrm{g}_{ab}^{(c_{\rm spin\,2})}$

as well. Based on our discussions before, we may check this by computing $\varepsilon_{abcd}\varphi^b\chi^c\mathcal{T}^d_e\,\chi^e$ after setting $c_{13} = 0$ in the expressions given in (5.40), (5.42) and (5.44) for the khronon's stress tensor components. Once more, a direct calculation shows $\varepsilon_{abcd}\varphi^b\chi^c\mathcal{T}^d_e\,\chi^e \neq 0$ in general, unless the following condition is imposed

$$(u\cdot\chi)\mathbf{\Pi}_a = c_{14}[(\nabla\cdot a)\boldsymbol{V}_a - (a\cdot\chi)a_a]\,, \tag{5.49}$$

with $\boldsymbol{V}_a$ defined as in (5.28). Thus, even though the circularity condition for the Killing vector φ^a is satisfied with respect to the spin-2 metric, the same is not necessarily true for the Killing vector χ^a unless the additional condition (5.49) is imposed. Consequently, even the hypersurface $\mathrm{g}^{(c_{\mathrm{spin}\,2})}_{ab}V^aV^b = 0$ is not necessarily a Killing horizon in a stationary, axisymmetric asymptotically flat spacetime with a universal horizon. As we have seen before, this could in principle be a problem, as the absence of a Killing horizon should give rise to deviations from GR that would have probably been observed by now. Also another question that could arise is that of falling back to GR in some decoupling limit scenario. On the other hand we don't have enough insight for the time being to be able to draw a definite conclusion on such matters, and therefore more work will be needed in order to understand better this case. It might be worth noting though that the solutions found so far all admit a Killing horizon outside the universal horizon, hence it seems likely that some condition ensuring this to happen should exist.

The above results clearly demonstrate that the circularity conditions (5.30) do not hold trivially in stationary, axisymmetric configurations in Hořava gravity. We may thus conclude that *in Hořava gravity, the mere assumptions of stationarity, axisymmetry and the existence of a universal horizon does not ensure the existence of a Killing horizon for any speed-c metric*. It might well be that some reasonable restriction on the foliation would be enough to satisfy the circularity conditions. We will not explore this possibility further here, as our intention was to simply demonstrate that circularity condition are not automatically satisfied and to motivate further work in this direction.

♦

Incredible as it might seem, black holes do exist in settings where Lorentz symmetry is broken and instantaneous propagation of signals in one frame is allowed. In this Chapter we have discussed extensively how such objects can be defined, both in the most general setting and in some specific cases with additional symmetries. In this last case, we were able to provide a local condition for the existence of black holes, and explore the possibility of having Killing horizons in additions to the universal horizon associated to the trapping of instantaneous signals, where such Killing horizons would lie with respect to the universal horizon and whether they would exist in the first place.

The last thing that we plan to discuss in this thesis, which will take up the next Chapter, is the dynamical formation of black holes in spacetimes with a preferred foliation. Indeed, until now we have been talking about black holes in generic foliated spacetimes, but without specifying any particular theory and thus without making use of any dynamical equation for such theory.

Clearly though, an important question for what concerns the "real world" is the actual dynamic of the formation of such objects, and therefore it will be interesting to try and find out how universal horizons can emerge from a gravitational collapse process.

6

SPHERICAL COLLAPSE AND THE INITIAL VALUE PROBLEM

Traditionally black hole solutions are studied in stationary spacetimes; in our work on black holes in Lorentz violating settings up to this point, after a generic discussion on the global properties of such objects, we also restricted to the stationary case. In general, the reason for doing so is that the presence of a Killing symmetry helps in identifying the event horizon. Also, without the need to take in account the dynamics of the theory but rather just a stationary state, the calculations normally become less complicated. Of course black hole solutions obtained in this way will represent eternal black holes, which exist since the beginning of time and will never disappear.

On the other hand, we know that the Universe is far from being stationary. For this reason, in order to have a more realistic idea of what black holes are, we will need to understand how they are formed and how they evolve. Much work has been put into trying to understand this problem. The first — and simplest — solutions that were found involve the collapse in spherical symmetry of thin massive shells or spheres of dust (this last case being usually referred to as *Oppenheimer-Snyder collapse*). Despite the insight these models gave into the formation of black holes, they are simplified models and hence more investigations were carried out. The first solutions for realistic black holes began to be found only recently, when numerical techniques and powerful computers allowed to numerically solve the Einstein equations to finally uncover realistic solutions for the collapse of matter into black holes and the dynamics of the resulting objects. Such investigations have reached an incredible level of precision, and are nowadays used to infer information on many phenomena that happen around black holes and that can potentially be related to available observations, thus being able to test the theories currently present on the market.

Of course the same ideas apply to modified gravity theories. If we were able to predict — at least numerically — the dynamics of the formation of black holes in theories different than GR we would be able to compare the difference between the predictions on observable quantities, this way hopefully being able to rule out some modified gravity candidates and thus converge towards a single theory of gravity. This is what we could ideally do with Hořava gravity (and Lorentz violating theories in general) in order to test the predictions of such theory.

The theory of stationary black holes in spacetimes with a preferred foliation has been laid down and discussed thoroughly in the previous Chapters, where we analysed the properties of such objects and we provided the conditions for their existence. At this point, it would be interesting to try and discuss the dynamics of the formation of black holes in this setting, in the hope to find some useful prediction to compare to observations.

Aside from the compaprison with observations, there is also a much more fundamental reason for studying dynamical collapse. The existence of black holes and universal horizons has been proved in a convincing way in the previous Chapters, and even before our work it was clear that such horizons can be found at least in sationary spacetimes. On the other hand there are some conjectures that this horizons could be unstable (Blas and Sibiryakov, 2011) and therefore, despite being possible to define, they might never form in a dynamical setting.

In order to answer these questions, the first thing we need is to discuss the dynamics of the theory in a spacetime with a preferred foliation. This is essentially what we will be doing in this Chapter. We won't be able to present here a complete tratment for the dynamics of gravitational collapse in Lorentz violating theories, which would require more time and expertise than what we have available, but we will try at least to derive the equations for the dynamics of such theories, which can then be used to study collapse problem. We will then analyse the dynamics of Hořava gravity and Einstein-Æther theory, pointing out the differences between the two theories; having done so, we will discuss the presence of the instantaneous mode in Hořava gravity, with the aim of discovering its origin and properties. Finally we will comment briefly on spherical collapse, and we will try to reinterprete some simulations in spherically symmetric Einstein-Æther theory, in the light of the new intuition we have on the theories at hand.

The hope is that this work will be useful to the community as a background study to refer to when running simulations — as generic as possible — of gravitational collapse in Lorentz violating theories.

6.1 the theories

In order to analyse the difference in the dynamics of the theories we are interested in, it would be convenient to be able to derive the equations of motion easily for both theories at the same time. The best way to do this is to employ the formalism for the covariant version of low-energy Hořava gravity. This will allow us to readily compare the differences in the dynamics of Einstein-Æther theory and Hořava gravity, since the action is formally the same, and so is for the most part the derivation of the equations of motion. In this Section we will therefore start from this form of the theory and derive the dynamics of the two theories under scrutiny, pointing out the differences between the theories when they occur. Also notice that, as discussed before in a different context, we will limit our discussion to the low-energy version of Hořava gravity. Indeed we are not interested in the high energy effects here, since for now we merely want to understand the dynamics of the formation of black holes; as discussed before in fact, the type of black holes that we expect to find in the Universe — which are in general quite large — can be safely considered low-energy phenomena despite representing the strong gravity regime.

We will start here with discussing Hořava gravity. As we have seen in Section 2.2.3, the low-energy version of Hořava gravity can be formulated as a scalar-tensor theory in a manifestly covariant manner (Jacobson, 2010); the tensor degree of freedom is represented as usual by the metric g_{ab} while the scalar degree of freedom is given by the scalar field T. The level surfaces of this scalar constitute the leaves of the preferred foliation. T is hence constrained to have a timelike gradient everywhere and one may introduce a timelike unit one-form u_a. Just as a reminder, the æther is defined as in (2.44) as

$$u_a = -N\nabla_a T \,, \tag{6.1}$$

and the theory is invariant under time reparametrisations

$$T \mapsto \tilde{T} = \tilde{T}(T) \,. \tag{6.2}$$

The action for the covariant low-energy version of Hořava gravity is then given as

$$\mathcal{S} = \frac{1}{16\pi G}\int \mathrm{d}^4x \, \sqrt{-g}\,(\mathcal{R} + \mathcal{L}) \,, \tag{6.3}$$

with $\mathcal{L}$ given in terms of the derivatives of the æther as

$$\mathcal{L} = -Z^{abcd}(\nabla_a u_c)(\nabla_b u_d) \,; \tag{6.4}$$

Z^{abcd} was defined in (2.28).

While, as we mentioned, we will only consider the low-energy part of Hořava gravity here, some remarks are in order. The terms that we are neglecting are higher-order in spatial derivatives, but they do not contain any time derivatives. This underscores the existence of a preferred foliation in Hořava gravity. Even though these terms can be written in a manifestly covariant way in the same fashion as the low-energy part of the action, in such a covariant formulation the full theory would appear highly fine-tuned, as higher-order time derivatives would have to cancel out (Sotiriou et al., 2011). Moreover, discarding the higher-order terms does not mean that the preferred foliation ceases to be preferred. As noted before, even in the low-energy theory that can be described in a covariant manner by action (6.3), T has to be nonzero and have a timelike gradient in *every* solution, thus implying that every solution comes with a special foliation. Additionally, action (6.3) actually contains more than two derivatives of T, which is an indication that the theory will not produce second order differential equations in a generic foliation.

Variation of action (6.3) (up to boundary terms) gives

$$\delta\mathcal{S} = \frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-\mathrm{g}}\left[E^{\mathrm{H}}_{ab}\,\delta\mathrm{g}^{ab} + 2(\nabla_a[N\text{Æ}^a])\delta T\right]\,, \tag{6.5}$$

where the definition of the vector field u_a given in (6.1) has been taken into account. Also notice that, even though we didn't add it explicitely in the action, we are nonetheless assuming the unit constraint $u^2 = -1$ to hold. In the present case, this constraint is equivalent to fixing the normalisation N in the definition of u_a.[1] The tensor E^{H}_{ab} is defined as

$$E^{\mathrm{H}}_{ab} = \mathcal{G}_{ab} - \mathcal{T}^{\mathrm{H}}_{ab}\,, \tag{6.6}$$

where $\mathcal{G}_{ab}$ is the (four-dimensional) Einstein tensor and $\mathcal{T}^{\mathrm{H}}_{ab}$ is the stress-energy tensor for the scalar field T. Æ^a is the functional derivative of the æther Lagrangian (6.4) with respect to the æther,

$$\text{Æ}^a \equiv \mathsf{p}^a{}_b\text{Æ}^b\,, \tag{6.7}$$

and p_{ab} is the projector onto the constant khronon leaves given by

$$\mathsf{p}_{ab} = \mathrm{g}_{ab} + u_a u_b\,; \tag{6.8}$$

remember that p_{ab} also acts as the induced metric on the leaves of the preferred foliation. Finally, Æ^a is manifestly orthogonal to the æther by construction. From (6.5), the equations of motion of Hořava gravity are

$$E^{\mathrm{H}}_{ab} = 0\,, \qquad \nabla_a[N\text{Æ}^a] = 0\,. \tag{6.9}$$

1 See also (2.45) for more details.

Æ^a already contains the second derivative of the æther, which implies that (6.9) contains third order time derivatives in an arbitrary foliation. However, the fact that Æ^a is orthogonal to the æther implies that only in the preferred foliation defined by T, the divergence in (6.9) is purely spatial and there are only two time derivatives (Jacobson, 2010).

Einstein-Æther theory (Jacobson and Mattingly, 2001), is a true vector-tensor theory. The fundamental fields are then the metric and the æther. As we saw before, the equations of motion of this theory can be derived from an action that is formally identical to (6.4), where the æther is constrained to satisfy only the unit norm constraint $u^2 = -1$ and is not necessarily hypersurface orthogonal. Variation with respect to the metric and the æther yields

$$\delta\mathcal{S} = \frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-\mathrm{g}}\left[E_{ab}\,\delta\mathrm{g}^{ab} + 2\text{Æ}^a\,\delta u_a\right] , \tag{6.10}$$

where the unit constraint has been imposed by constraining the æther's variation. E_{ab} is defined as

$$E_{ab} = \mathcal{G}_{ab} - \mathcal{T}_{ab} , \tag{6.11}$$

with $\mathcal{T}_{ab}$ being the stress energy tensor of the æther given explicitely in (2.32). From (6.10), the equations of motion of Einstein-Æther theory are

$$E_{ab} = 0 , \qquad \text{Æ}^a = 0 . \tag{6.12}$$

$\mathcal{T}_{ab}$ and $\mathcal{T}^{\mathrm{H}}_{ab}$ are formally identical as they come from formally identical actions under variation with respect to the metric. This means that, if one were to impose the hypersurface orthogonality condition $u_a = -N\nabla_a T$ on the æther as an additional simplifying assumption at the level of the equations of motion in Einstein-Æther theory, the systems of equations (6.9) and (6.12) will have the same Einstein equations. Moreover, any such hypersurface-orthogonal solution of Einstein-Æther theory will also be a solution of Hořava gravity. While the converse is not generically true, it was shown to hold for spherically symmetric, asymptotically flat solutions under the assumption that all leaves of the foliation reach the centre and the centre is regular (Blas et al., 2011). It was also shown to hold for static, spherically symmetric, asymptotically flat solutions without any further assumptions (Barausse and Sotiriou, 2012), as well as for static, spherically symmetric solutions with more general asymptotics but with a regular universal horizon (Bhattacharyya and Mattingly, 2014).

6.2 DYNAMICS OF THE THEORIES

At this point, we are ready to discuss the difference in the dynamics of the two theories. We have already seen in Chapter 2 that Einstein-Æther theory propagates a spin-1 and spin-0 mode, in addition to the usual spin-2 graviton, while Hořava gravity — due to the hypersurface orthogonality of u_a (6.1) — only propagates a spin-0 mode. This is surely a quite evident difference in the dynamics of the two theories.

Here, however, we wish to focus on the more subtle differences that are not related to the existence of vector modes. To this end, we wish to compare the non-perturbative dynamics of Hořava gravity with that of Einstein-Æther theory when the æther is constrained to be hypersurface orthogonal — at the level of the equations — throughout the evolution.[2] An important subcase that we will discuss later in more detail is spherically symmetric collapse, since spherically symmetric vectors are naturally hypersurfance orthogonal. However, in this Section we will opt to be as general as possible, and we will not assume any symmetries.

2 Note that, in general, evolution could generate vorticity, so our condition is stronger that selecting vorticity-free initial data. It might well be that constraining the aether to be hypersurface orthogonal throughout the evolution could lead to an overconstrained system in the absence of extra symmetries, e.g. spherical symmetry. This is an interesting open question, but it will not concern us here, as we simply seek for the most general setting in which we can straightforwardly compare Einstein-Æther theory and Hořava gravity.

By virtue of Frobenius's theorem (Wald, 1984), a hypersurface orthogonal (hence twist-free) unit timelike æther satisfies

$$\nabla_a u_b - \nabla_b u_a = -u_a a_b + u_b a_a \, , \tag{6.13}$$

where a_a is the acceleration of the æther congruence defined as usual as

$$a^a = \nabla_u u^a \qquad \Leftrightarrow \qquad a_a = N^{-1} \boldsymbol{\nabla}_a N \, . \tag{6.14}$$

Here $\boldsymbol{\nabla}_a$ is the covariant derivative projected on the preferred foliation, and the second expression in (6.14) above is a consequence of the definition of u_a together with (6.13). One may thus expand the covariant derivative of the æther as

$$\nabla_a u_b = -u_a a_b + K_{ab} \, , \tag{6.15}$$

where K_{ab} is the extrinsic curvature of the leaves of the foliation, given by

$$K_{ab} = \frac{1}{2} \pounds_u \mathsf{p}_{ab} \, , \tag{6.16}$$

and is purely spatial by definition. The mean curvature K, which is the trace of the extrinsic curvature, is given by

$$K = \mathsf{g}^{ab} K_{ab} = \mathsf{p}^{ab} K_{ab} = (\nabla \cdot u) \, . \tag{6.17}$$

We may now use the above quantities and relations to adapt the equations of motion of both theories, (6.9) and (6.12), to the foliation defined by the æther — or alternatively by T. For Hořava gravity this is imperative, as mentioned earlier, for it is only in this foliation that the equations become second order in time derivatives. For Einstein-Æther theory, on the other hand, this is simply a choice we make in order to facilitate the comparison with Hořava gravity. It is also worth noting that, even though we are adopting a particular foliation, we will refrain for the time being from adopting any specific coordinate system.

As we already discussed, for a hypersurface orthogonal æther the Einstein equations in both the theories are formally identical. One may furthermore show (Barausse et al., 2011; Jacobson, 2011) that the covariantized Bianchi identities are formally identical as well. When adapted to the preferred foliation, the generalised Bianchi identities for both theories read

$$\begin{aligned} \partial_T E^T{}_i &= 0\,, \\ \partial_T E^T{}_T + (\sqrt{-\mathrm{g}})^{-1}\partial_i[\sqrt{-\mathrm{g}}N\text{Æ}^i] &= 0\,, \end{aligned} \tag{6.18}$$

where $i = \{1,2,3\}$ denote *coordinate indices* on the preferred leaves, E_{ab} is either E^{H}_{ab} as found in (6.6) or E_{ab} as found in (6.11) — for a hypersurface orthogonal æther — and in (6.18) it was assumed that all Einstein's equations (but not the æther/T equations) are satisfied *on the given leaf*. Note that a $(1+3)$ decomposition of the equations of motion of Einstein-Æther theory need not necessarily be performed with respect to the æther's foliation and, in general, the corresponding constraint equations are a combination of Einstein equations and the æther's equation of motion. However, when formulated as a theory of a one-form, the constraint equations of Einstein-Æther theory adapted to the æther's foliation do not involve the æther's equations of motion but only the Einstein equations in the form of $(E)^T{}_T = (E)^T{}_i = 0$. Thus according to (6.18), these constraints are also preserved in time once the æther becomes 'on shell' (6.12) as well. For Hořava gravity, on the other hand, the only foliation where a $(1+3)$ decomposition of the equations of motion produces sensible results is the preferred foliation; only then proper constraint equations can be found in the form of $(E^{\mathrm{H}})^T{}_T = (E^{\mathrm{H}})^T{}_i = 0$ which are first order in the T-derivative and according to (6.18) are preserved in time once the khronon becomes 'on-shell' (6.9).[3]

In both theories, the constraint $E^T{}_T = 0$ constitutes the *energy constraint equation* explicitly given by

$$(1-c_{13})K_{ab}K^{ab} - (1+c_2)K^2 + c_{14}\left[2(\boldsymbol{\nabla}\cdot a) + a^2\right] - \mathrm{R} = 0\,, \tag{6.19}$$

3 A decomposition with respect to a different foliation is of course possible, but the equations of motion will be higher order in time derivatives and the constraints produced will be extremely cumbersome to treat. For this reason we limit ourselves to the preferred foliation.

where R is the intrinsic scalar curvature of the leaves of the foliation. The equations ${E^T}_i = 0$ represent the *momentum constraint equations* and take the form

$$(1 - c_{13})\boldsymbol{\nabla}_c {K^c}_a = (1 + c_2)\boldsymbol{\nabla}_a K \, . \tag{6.20}$$

The remaining Einstein's equations, i.e. those completely projected onto the preferred foliation, reduce for both theories to an *evolution equation* for the mean curvature given by

$$c_\ell \nabla_u K = -(1 - c_{13})K_{ab}K^{ab} - \frac{c_{123}}{2}K^2 + \left[1 - \frac{c_{14}}{2}\right](\boldsymbol{\nabla} \cdot a + a^2) \, , \tag{6.21}$$

where $c_{123} = c_2 + c_{13}$ and $c_\ell = 1 + (1/2)c_{13} + (3/2)c_2$, and an *evolution equation* for the traceless part $[K]_{ab} = K_{ab} - (K/3)\mathsf{p}_{ab}$ given by

$$\begin{aligned} \pounds_u [K]_{ab} = {} & 2[K]_{ac}{[K]_b}^c - \frac{K}{3}[K]_{ab} + \frac{1 - c_{14}}{1 - c_{13}}\left[a_a a_b - \frac{a^2}{3}\,\mathsf{p}_{ab}\right] \\ & + \frac{1}{1 - c_{13}}\left[\boldsymbol{\nabla}_a a_b - \frac{(\boldsymbol{\nabla} \cdot a)}{3}\,\mathsf{p}_{ab} - \mathsf{R}_{ab} + \frac{\mathsf{R}}{3}\,\mathsf{p}_{ab}\right] , \end{aligned} \tag{6.22}$$

where R_{ab} is the Ricci curvature of the preferred hypersurfaces. Collectively, the equations in (6.19), (6.20), (6.21), and (6.22) provide all Einstein equations for both the theories.

As noted above, (6.21) and (6.22) provide a set of evolution equations for the extrinsic curvature that are first order in time derivatives with respect to the preferred foliation in both the theories (for Einstein-Æther theory, these equations were already obtained in Garfinkle et al. (2007)). Taken together with (6.16), these provide a set of first order evolution equations for the pair of conjugate variables consisting of the components of the induced metric and those of the extrinsic curvature. To turn these equations explicitly into a set of coupled partial differential equations, one needs to introduce a set of coordinates on the leaves of the hypersurfaces and perform a lapse-shift decomposition of the metric. However, this goes beyond our current goal; we merely wish to point out here that even in the most general setting the 'metric-extrinsic curvature pair' can be evolved in the same manner with respect to the preferred foliation in both theories. The difference between the dynamics of the theories — and the related issue of the existence of the instanteneous mode in Hořava theory — stems from the evolution of the æther/T. We will take up this issue next and study it in more detail in spherical symmetry.

6.3 THE INSTANTANEOUS MODE

In terms of the kinematic variables introduced previously, the quantity Æ_a (6.7) is given by

$$\text{Æ}_a = \frac{c_{123}}{(1-c_{13})}\boldsymbol{\nabla}_a K - c_{14}[K a_a + \pounds_u a_a - 2K_{ac}a^c] \,. \tag{6.23}$$

This allows one to interprete the æther's equation of motion (6.12) in Einstein-Æther theory as an evolution equation for the acceleration (Garfinkle et al., 2007) as

$$\pounds_u a_a = 2K_{ac}a^c - K a_a + \frac{c_{123}}{c_{14}(1-c_{13})}\boldsymbol{\nabla}_a K \,. \tag{6.24}$$

Needless to say the above is not satisfied, in general, in Hořava gravity; the 'khronon's equation of motion' in Hořava gravity is given instead by (6.9)

$$\nabla_a[N\text{Æ}^a] = 0 \qquad \Leftrightarrow \qquad \boldsymbol{\nabla}_a[N^2\text{Æ}^a] = 0 \,, \tag{6.25}$$

where the expression for Æ^a is identical to that given in (6.23). The difference in the dynamics in the two theories thus lies in the difference between the nature of (6.24) and (6.25). To study them closely, in the following we will restrict ourselves to spherical symmetry, which will allow us to easily integrate (6.25). Note that in spherical symmetry, the hypersurface orthogonality of the æther is guaranteed kinematically, without the need to impose it as an additional constraint.

Toward setting up a suitable coordinate system that makes the spherical symmetry manifest, let us start with some basic observations: in any coordinate system adapted to the æther's foliation, the time coordinate is identical to T, and hence subject to the reparametrisations given by $T \mapsto \tilde{T} = \tilde{T}(T)$. Next, the unit spacelike vector s_a along the acceleration, defined by

$$a_a = (a \cdot s)s_a \,, \qquad (s \cdot s) = 1 \,, \qquad (u \cdot s) = 0 \,, \tag{6.26}$$

defines a natural spacelike direction orthogonal to the angular directions by virtue of spherical symmetry. In order to be as general as possible — and in particular, to make our subsequent conclusions manifestly independent of any 'gauge choices' — we will introduce a coordinate system adapted to the preferred foliation consisting of the 'time coordinate' T and a 'radial coordinate' R in which (along with the definition of u_a) we have

$$\begin{aligned} u^a &= N^{-1}\partial_T - N^{-1}N^R\partial_R \,, \\ s^a &= S^{-1}\partial_R \,, \\ s_a &= SN^R\nabla_a T + S\nabla_a R \,, \end{aligned} \tag{6.27}$$

such that the functions $N = N(T, R)$, $S = S(T, R)$, and $N^R = N^R(T, R)$ completely describe the æther configuration in a manifestly spherically symmetric manner. Note that for the above choice of coordinates, the shift vector is $\boldsymbol{N}^a = N^R\partial_R$. Furthermore, the projector (6.8) can be written as

$$\mathsf{p}_{ab} = s_a s_b + \hat{\mathsf{g}}_{ab} \,, \tag{6.28}$$

where $\hat{\mathsf{g}}_{ab}$ is the metric on a unit two-sphere up to a conformal factor represented by the areal radius r squared

$$r = \sqrt{\frac{\text{Area of two-sphere}}{4\pi}} \,, \qquad \hat{\mathsf{g}}_{ab} = r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \,,$$

and θ and ϕ are the usual polar coordinates on the unit two-sphere. In what follows, r will not be treated as a coordinate. Rather, the coordinate system we have constructed consists of the coordinate variables $\{T, R, \theta, \phi\}$, and the areal radius is given as $r = r(T, R)$, in the same way as N, S, and N^R. Thus in the present coordinate system the full metric is given by

$$\mathsf{g}_{ab} = -N^2\mathrm{d}T^2 + S^2(N^R\mathrm{d}T + \mathrm{d}R)^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \,; \tag{6.29}$$

the æther and the metric are completely specified by the four functions $N(T,R)$, $N^R(T,R)$, $S(T,R)$ and $r(T,R)$. Hence, the equations of motion of the two theories along with some suitable gauge choice will allow us to solve for these functions.

We may now integrate the equation of motion for T in Hořava gravity (6.25) to obtain

$$\partial_R \left[r^2 S N^2 Æ^R \right] = 0 \qquad \Leftrightarrow \qquad (s \cdot Æ) = \frac{f_{\text{IM}}(T)}{r^2 N^2} \,, \tag{6.30}$$

where $f_{\text{IM}}(T)$ is some integration constant. Plugging this into the expression (6.23) of $Æ_a$ we then end up with a first order evolution equation for the acceleration very similar to (6.24)

$$\partial_T (a \cdot s) = N^R \partial_R (a \cdot s) - N\hat{K}(a \cdot s) + \frac{c_{123} N \partial_R K}{c_{14}(1 - c_{13})S} - \frac{f_{\text{IM}}(T)}{c_{14} r^2 N}, \tag{6.31}$$

where $\hat{K} = \hat{g}^{ab} K_{ab}$. This equation contains, in the most explicit manner, the most crucial difference between the dynamics of Einstein-Æther and Hořava theories. Indeed, one obtains the æther's equation of motion in Einstein-Æther theory (6.24) upon setting $f_{\text{IM}}(T) = 0$ *for all* T, while $f_{\text{IM}}(T) \neq 0$ characterises those solutions of Hořava theory which are *not* solutions of Einstein-Æther theory. Finally, as soon as one solves for $(a \cdot s)$ using (6.31), one may solve for the lapse by integrating (6.14) on a given T slice, thus obtaining

$$\partial_R \log N = (a \cdot s) S \,, \tag{6.32}$$

which implies

$$\log N(T,R) = \log N(T,\infty) + \int_\infty^R \mathrm{d}R' (a \cdot s) S(T,R') \,. \tag{6.33}$$

In this way, all the relevant functions determining the spacetime-æther/T configuration in both the theories can be solved for.

In both theories there is still the reparametrisation freedom for T. In Hořava gravity, this is a symmetry of the theory itself, whereas in Einstein-Æther theory it comes as a consequence of our restriction that the æther be hypersurface orthogonal. Such reparametrisation freedom for T implies that we can also reparametrise N as $N \mapsto \tilde{N} = (\mathrm{d}\tilde{T}/\mathrm{d}T)^{-1}N$; this implies that $\log N$ picks up a function of T *additively* under the aforementioned reparametrisations. We can choose then to reparametrise T in such a way that

$$\log N(T, \infty) = 0 \,, \tag{6.34}$$

which was the choice made in Garfinkle et al. (2007).

From the reasoning we just introduced, it becomes quite clear that indeed generic Hořava gravity solutions are characterized by a non-zero $f_{\text{IM}}(T)$ while $f_{\text{IM}}(T) = 0$ in Einstein-Æther theory. The easiest way to see this, is to consider the equation for the æther of Einstein-Æther theory (6.12) in (6.30): in this case, the integration constant $f_{\text{IM}}(T)$ has to be null, for the equation to make sense. On the other hand, we are considering spherical symmetry, and until now we have claimed that the solutions for both theories should be the same. It seems though to now that there is indeed a difference, represented by the presence of the integration constant, rendering the solutions different. What we are left to do then is to understand this difference.

The root of such difference between the two theories is in the following: turning the T equation into an evolution equation for the lapse N in Hořava gravity involves integrating a divergence on each slice.[4] Hence there is an elliptic part in this system of equations that is absent in Einstein-Æther theory. It should be stressed that this elliptic part is fundamentally different from the constraint equations, even though the latter are also elliptic. The main difference has to do with the fact that constraints are preserved by time evolution and hence need to be imposed only on an initial slice, while the divergence in (6.30) has to be integrated on *every slice* and $f_{\text{IM}}(T)$ is to be determined by suitable boundary/asymptotic conditions. This will be discussed in more detail below. For instance, for generic functional forms of $f_{\text{IM}}(T)$, (6.31) is singular if either $r = 0$ or $N = 0$. Thus, the physical

4 Recall that we are working in the preferred foliation, so the lapse N cannot be set to a constant by making a gauge choice, and hence it should be determined by the field equations.

requirement of regularity at the centre or on a universal horizon where $N = 0$ for our choice of T can impose $f_{\text{IM}}(T) = 0$.

This is simply the non-perturbative manifestation of the instantaneous mode discussed in Blas and Sibiryakov (2011) in a perturbative setting. Indeed, when spherical perturbations around a black hole were considered in Blas and Sibiryakov (2011), it was the assumption of regularity on the universal horizon that forced the instantaneous mode to vanish, in agreement with what has been mentioned above.

The above conclusions can be generalized beyond spherical symmetry, albeit somewhat qualitatively. To that end, we may begin by recalling that in diffeomorphism invariant scalar-tensor theories the equation determining the scalar field is dynamically redundant, as it can be obtained by taking a divergence of the field equations for the metric. Hence, one can in principle solve the latter only and neglect the scalar's equation altogether. Since Hořava gravity can be written as a diffeomorphism invariant scalar-tensor theory, one can apply this logic. This then implies that consistent solutions can be obtained by solving only (6.19)-(6.22) (where we have conveniently neglected (6.9) only after forming the constraint equations). Equation (6.19) can then be turned into the following Poisson type elliptic equation for ϱ, defined through $N = \varrho^2$:

$$\boldsymbol{\nabla}^2\varrho = \frac{\varrho}{4c_{14}}\left[\mathsf{R} - (1-c_{13})K_{ab}K^{ab} + (1+c_2)K^2\right] \,. \tag{6.35}$$

As already pointed out in Donnelly and Jacobson (2011), this equation allows one to solve for the lapse N *on each slice of the preferred foliation*. One can then subsequently compute the acceleration from (6.14). Thus, (6.35), (6.20), (6.21), and (6.22) provide a complete set of equations that can dynamically determine the spacetime and the foliation in Hořava gravity.

Since (6.35) is a second-order *elliptic* equation in ϱ that is not preserved by time evolution when T is not taken to be on-shell [see also (6.18)], it is indeed expected that its solution should depend on two integration constants — which are actually functions of the preferred time T. This matches precisely the result we obtained previously in spherical symmetry. While one of these functions of T can be set to a desired value by the (yet-to-be-fixed) reparametrisation freedom of T, the second one will be related to the instantaneous mode of the theory, analogous to the function $f_{\mathrm{IM}}(T)$ introduced above, and cannot be done away with even after fixing said reparametrisation freedom.

The above logic does not apply to Einstein-Æther theory, simply because the æther equation is not dynamically redundant even when the æther is hypersurface orthogonal. Indeed, solutions of (6.35), which obviously also hold in Einstein-Æther theory, do not always satisfy the æther's equation of motion (6.24).

Though slightly less rigorous than our spherically symmetric treatment, this last analysis has two advantages: it is more general and it clearly demonstrates that in Hořava gravity and in the preferred foliation the equations can be thought of as a system of an elliptic equation that needs to be imposed on every slice, elliptic equations that are preserved by time evolution and hence constitute constraints, and dynamical equations that generate the spacetime together with its foliation. Donnelly and Jacobson (2011) reached the same conclusion by means of a Hamiltonian analysis of the theory.

6.4 SPHERICAL COLLAPSE

The problem of spherically symmetric collapse provides one of the simplest settings to which the preceding analysis can be directly applied, thereby allowing us to compare the dynamics of the two theories explicitly. Spherical collapse in Einstein-Æther theory has been in fact studied in Garfinkle et al. (2007), while an analogous simulation in Hořava theory is yet to be performed.[5] In light of the relation between evolution in Einstein-Æther the-

5 See, however, Saravani et al. (2014) where cuscuton theory is used as a proxy for Hořava gravity.

ory and Hořava gravity discussed in the previous Sections, it is tempting to revisit the results of Garfinkle et al. (2007), potentially reinterpreting some of them, and to attempt to draw some general conclusions about spherical collapse in Hořava gravity.

To be more specific, in Garfinkle et al. (2007) spherically symmetric collapse in Einstein-Æther theory with a minimally coupled scalar field ψ was studied; ψ represents here a shell of collapsing matter. The evolution of the system was performed by adapting (6.12) to the foliation described by the æther, possibility granted by the æther being hypersurface orthogonal due to spherical symmetry. This foliation, as pointed out earlier, is equivalent to the preferred foliation of Hořava gravity. The equations in (6.12) were then supplemented with appropriate equations of motion for ψ.

Simulations were performed for two different values for the speed s_0 of the spin-0 mode. In the first case the couplings c_3 and c_4 were set to zero, and the remaining two parameters of the theory, c_1 and c_2, were chosen such that s_0 was set to unity, i.e. equal to the speed of light. Two values of c_1 were considered. For $c_1 = 0.7$ a regular (Killing) horizon forms as a result of the collapse while for $c_1 = 0.8$ no such horizon seems to form and '... the evolution seems to become singular, thus indicating the formation of a naked singularity' (Garfinkle et al., 2007). The main reason for considering the specific values of the c_i parameters and s_0 was related to the fact that no static solutions had been found for the same values and $c_1 \geqslant 0.8$ in Eling and Jacobson (2006). Indeed, such result was interpreted as verifying the absence of black holes for these parameters. However, static black holes were later found for that very same choice of the couplings in Barausse et al. (2011), and it was argued there that the only reason these solutions were not found in Eling and Jacobson (2006) was an insufficient accuracy in the numerics performed there. This puzzling situation definitely deserves further investigation. However, these simulations are not presented in a sufficiently detailed way in Garfinkle et al. (2007), and so it is hard to interpret them in light of the later results of Barausse et al. (2011) or our analysis of the previous Sections. For this reason, we will not consider them further here.

The second set of parameters was chosen such that the speed of the spin-0 mode was set to $\sqrt{2}$. With suitably chosen initial conditions, evolution led to the formation of a *regular* spin-0 horizon *inside* the metric horizon. Furthermore, at sufficiently 'late times,' the geometry *outside the spin-*0 *horizon* settled down to the static solutions of Eling and Jacobson (2006) to high accuracy.[6] Moreover, the simulations revealed that the preferred frame lapse function N 'is driven to zero as the singularity is approached' (Garfinkle et al., 2007).

A vanishing of the lapse function at any given point of an evolution simulation in a gravity theory is strongly indicative of a breakdown of the corresponding foliation. A well known example of this is the study of spherically symmetric collapse in general relativity in Schwarzschild coordinates, where a similar situation is expected toward the formation of the Killing horizon. On the other hand, provided one can be certain about the horizon-crossing properties of a certain foliation, forcing the lapse to vanish asymptotically in time and as the singularity is approached can have some advantages from a numerical perspective. Since studying evolution with respect to the foliation defined by the æther is merely a choice in Einstein-Æther theory, determining whether this is the optimal choice is a point that deserves further discussion.

The æther's foliation in spherical symmetry will penetrate all Killing horizons, as the latter are null surfaces and the aether is always timelike. Considering also its privileged status in Einstein-Æther theory, it was certainly a natural choice for Garfinkle et al. (2007). One of the goals of Garfinkle et al. (2007) was indeed to verify whether regular spin-0 horizons would emerge from spherical collapse in Einstein-Æther theory. Nonetheless, this foliation is special, and there is a way in which using it in this setting resembles us-

6 Note that Garfinkle et al. (2007) predates Barausse et al. (2011), and thus were only able to compare their results with Eling and Jacobson (2006).

ing Schwarzschild coordinates in spherically symmetric collapse in general relativity: it does not penetrate the universal horizon.

Indeed, the vanishing of the lapse function N in the preferred foliation can have an alternative interpretation as an asymptotic formation of a *universal horizon*. In a static and spherically symmetric geometry, a universal horizon (Barausse et al., 2011; Blas and Sibiryakov, 2011) is a leaf of the preferred foliation that is also a constant r hypersurface (and hence a hypersurface generated by the Killing vector associated with staticity), turning it into an event horizon even for arbitrarily fast propagations (Bhattacharyya et al., 2016b). In particular, the fact that a universal horizon is generated by a Killing vector implies that the preferred frame lapse function, subjected to the boundary condition (6.34), will also vanish on the universal horizon. Moreover, a universal horizon can only occur in the asymptotic future in the preferred time. These observations, along with the fact that the geometry outside the horizon settles down to the appropriate static, and essentially unique solution (Eling and Jacobson, 2006; Barausse et al., 2011; Blas and Sibiryakov, 2011; Bhattacharyya and Mattingly, 2014) strongly suggest that the simulations of Garfinkle et al. (2007) revealed the asymptotic formation of a universal horizon in the 'late time' phase. The notion of a universal horizon was introduced several years after the work reported in Garfinkle et al. (2007) appeared though, and it is therefore natural that the above interpretation escaped its authors.

In situations where a universal horizon may form, working in the preferred foliation is clearly not the optimal choice. The simulation will indeed inevitably stop as the universal horizon is approached, and one will never be able to cross it in this setup. If the simulations of Garfinkle et al. (2007) were to be performed again in a different foliation, it seems likely that one would be able to trace the formation and evolution of the universal horizon and verify whether the result leads to the static solutions of Barausse et al. (2011), all the way to the universal horizon and beyond.

We now turn our attention to what the simulation of Garfinkle et al. (2007) can teach us about spherical collapse in Hořava gravity. Taking into account the connection between Hořava gravity and Einstein-Æther theory as discussed in detail in the previous Sections, spherical collapse in the latter will be identical to spherical collapse in the former provided boundary conditions that set $f_{\text{IM}}(T) = 0$ in (6.25) have been chosen. The suitable boundary condition is simply regularity at the origin, $r = 0$ (up to the formation of the singularity and/or universal horizon). Note that using the preferred foliation is not a choice but a necessity in Hořava gravity. Hence, the fact that the evolution seemingly 'ends' with an asymptotic formation of a universal horizon appears to be a confirmation of the claim that the universal horizon is also a Cauchy horizon in theories like Hořava gravity as we discussed in Chapter 5 (see also Blas and Sibiryakov, 2011; Bhattacharyya et al., 2016b), where the preferred foliation actually determines the causal structure and further boundary data are required to determine the evolution.

♦

In whichever theory of gravity we decide to work with, black holes are among the most important objects to take in consideration. Hence, studying the properties of such objects and, in particular, the details of their dynamical formation and development can provide a wealth of information. In the first place, as we discussed above, it can shed some light on whether universal horizons are indeed formed in a dynamical setting, or whether they can just be defined mathematically and in stationary spacetimes, but don't actually emerge from a collapse scenario. Additionally, it can provide a number of potentially observable effects which could then be used in conjunction with observational data to study the viability of different theories.

If we choose to consider Lorentz violating theories, the study of these kinds of solutions becomes even more important. In fact, as has been pointed out many times throughout this thesis, when considering Lorentz violating theories black holes might not even exist. Closer inspection though reveals that black holes do in fact exist, and present us with a number of surprisingly interesting properties; Chapter 5 contains a quite extensive discussion of such properties, so we won't repeat them here. The question that comes naturally to mind at this point is then that of the dynamics of the formation of such black holes. The present Chapter has been dedicated to a preliminary study of this problem.

In the first place, we derived the Initial Value Problem (IVP) for spacetimes with a preferred foliation, and we discussed the differences in the dynamics of Hořava gravity and Einstein-Æther theory. Having done so, we dedicated a Section to discussing the presence of the instantaneous mode in Hořava gravity, as opposed to Einstein-Æther theory, and to show where it originates. Finally we discussed some previous results about spherical collapse, reinterpreting them to argue that the emergence of an universal horizon in numerical codes might have been already uncovered a decade or so ago.

Having set up the framework for studying the IVP in Lorentz violating theories, notably in Hořava gravity, we hope that this work will be useful in the future as a background to refer to when running numerical simulations of collapse in the settings we discussed. In particular one question that would be interesting to answer is what happens to the collapse simulations in Einstein-Æther theory when a foliation different from the æther foliation is used. Also, it would be interesting to study the problem of collapse in less symmetric settings in Hořava gravity, in order to understand if the solutions of such collapse actually settle down to the solutions we discussed in the previous Chapters.

7

CONCLUSIONS

Throughout this thesis, we tried to uncover some characteristics of one of the most controversial proposals for a UV complete theory of gravity: Hořava gravity. This theory provides a potentially renormalisable theory of gravity, at the price of abandoning Lorentz symmetry as a fundamental symmetry. In fact the way Hořava gravity manages to ensure renormalisability is by introducing higher order dispertion relations, which in turn lead to superluminal and potentially istantaneous propagation; in order to do this though Lorentz symmetry is explicitly broken. Lorentz violations can be then brought under experimental constraint at low energies, which is in the regime where Lorentz symmetry is very well tested; Lorentz symmetry itself ceases nonetheless to be a fundamental symmetry of nature. On the other hand many aspects of the theory are still unclear, and our broad goal was to discuss them and hopefully shed some light over the most obscure corners of the theory.

Three main problems are encountered when dealing with Lorentz breaking theories. The first — which we didn't touch upon — regards the actual renormalisability of the theory. While there are indications that the theory is renormalisable, notably a first proof of renormalisability at all orders for projectable Hořava gravity (Barvinsky et al., 2016), a formal proof is still needed in the general case. The second issue is that of coupling in a meaningful way the Lorentz symmetric matter sector to a Lorentz violating gravity sector. The third problem is instead that of the existence of black holes in the theory. The last two problems are the ones we concentrated upon in the bulk of this work.

After a rather generic introduction on the problem of quantum gravity and on the potential solution offered by Lorentz violating theories, Chapter 3 studied the issue of meaningfully coupling matter to Lorentz violating gravity. While often disregarded, this problem is of fundamental importance for the theory. The issue lies in the fact that, since Lorentz symmetry in the matter sector is constrained to a very high precision, there needs to be a way to avoid Lorentz violations from the gravity sector to percolate to matter through graviton loops. In particular, in our work we considered a model proposed a few years ago in Pospelov and Shang (2012) which noticed that the cunundrum is solved when the two sectors are coupled through power suppressed interactions; this suppression allows to regulate the percolation of Lorentz violations to matter by introducing a sufficient separation between the scale where Lorentz violations enter the game in the gravity sector and the Plank scale. While studying this model though, some naturalness issues were uncovered in the vector sector of the theory: this last problem was potentially avoided through the addition of a single term to the action. The interesting feature of this term is that of mixing time and space derivatives; this also makes this term of higher order. Our work concentrated on checking whether adding more generic terms of the type considered in Pospelov and Shang (2012) can harm the theory from the renormalisability and stability point of view. What we discovered is that, while the theory remains well behaved as far as renormalisability is concerned, problems arise when studying stability. In fact the theory with the most generic mixed derivative terms features two scalar degrees of freedom, as opposed to the single scalar degree of traditional Hořava gravity. Worse than that, one of these modes turned out to be ill-behaved since it develops a tachionic instability; such an instability was shown to be present at all energies, and the only thing that we can do to it is to trade it for a ghost degree of freedom. This possibility hence doesn't help in solving the problem. More work is required therefore to investigate the possibility of eliminating this disturbing degree of freedom, or potentially to investigate the possibility of abandoning the mixed derivative extension altogether and solve the naturalness problem in some alternative way.

The second topic we discussed, in the last three Chapters of this work, is that of black holes. Indeed, the problem with the definition of black holes is even more delicate. If a theory allows instantaneous propagation of signals, the very concept of black hole could in fact be at risk; the reason for this is that the "traditional" definition of black hole relies so heavily on the local causal structure defined by Lorentz symmetry that it seems unlikely that this kind of objects could exist once we give up Lorentz symmetry. This is a quite involved discussion, and was therefore covered over the course of a few Chapters.

First, we studied the causal structure of spacetimes with a preferred foliation. This is important because these are the spacetimes which incarnate the most generic setting that shares the features we are interested in when it comes to Hořava gravity. At the same time, employing such a generic setting allows us to not be restricted to a particular theory but to draw conclusions which can be potentially applied to a range of theories with similar characteristics. Having defined all the formalism related to the causal structure and the asymptotic behaviour of foliated spacetimes in Chapter 4, we then went on in Chapter 5 to rigorously define black holes and universal horizons in the settings described above; one of the most interesting results we found is a confirmation to the conjecture which claimed that universal horizons are Cauchy horizons.

Having defined black holes and horizons in the most general way possible, we concentrated on some particular cases with added symmetries (such as stationary and axisymmetric spacetimes) in order to better understand some of the characteristics of the theories at hand. We uncovered a number of interesting properties of universal horizons and Killing vectors in spacetimes with a preferred foliation; the most important result was arguably the local characterisation of universal horizons in stationary spacetimes — given by the hypersurface on which $(u \cdot \chi) = 0$.

To finish, in the same Chapter we proved that, in an axisymmetric spacetime, if Killing horizons exist they always lie outside the universal horizons and therefore hide the universal horizon to an observer outside the Killing horizon. This might seem something that does not have too much bearing to the question at hand, but it's actually a quite important result. Indeed, if there was to be a universal horizon that was not hidden behind a Killing horizon, the structure of the black hole spacetime would be quite different to the usual general relativistic case, starting with the very location and radius of the horizon; such a difference would have likely been already found. In addition we tried to provide some conditions for the existence of said Killing horizons in the case of stationary axisymmetric Hořava gravity. Indeed, in general relativity the existence of Killing horizons is insured by the so called *energy conditions*. Since in the case of Hořava gravity the energy conditions don't necessarily hold, we worked out two conditions that can guarantee the formation of Killing horizons in Hořava gravity as well. The physical interpretation of the conditions we found though is not entirely clear to us at the present time, and therefore we don't know whether such conditions can be identified with some new energy conditions or if they are simply accidental.

Up to this point, we have clearly established the existence of black holes along with a wealth of the properties that such objects show. Most of this — with the notable exception of the generic global definitions which do not require any restriction — was studied in stationary spacetimes, and therefore the discussion was centered around eternal black holes. On the other hand, it would be really interesting to also study how black holes are formed. To do so, and hence with the broad goal of a thorough study of gravitational collapse in spacetimes with a preferred foliation, we needed to establish the initial value problem (IVP) in such a setting. This is what was done in Chapter 6; there we first analysed the dynamics of Hořava gravity and Einstein-Æther theory, highlighting the differences between the two. Having done so, we moved on to the spherically symmetric case where we reinterpreted some previous results obtained from numerical simulations of Einstein-Æther theory. In fact, it seems that numerical evidence for the formation of an universal horizon was already uncovered. This fact though was not noticed, due to the limited knowledge at the time of the properties of this type of horizon. Newer simulations are thus required in order to establish this result with certainty.

7.1 FUTURE WORK

In this work we hopefully managed to shed light on a good deal of dark spots that still endured in the contest of Lorentz violating theories and, for the first time, we provided rigorous definitions to many concepts that were qualitatively known but not quite rigorously formalised until now. On the other hand, it would be an overstatement to claim that we gave a complete and final solution to most of the problems we touched upon. Our results provide a minor, albeit important, contribution to the field. Many things still require the community's attention in order to be clarified completely.

First, a final proof of the renormalisability of Hořava gravity is required. As we have seen throughout the thesis, there are indications that the theory is indeed renormalisable, but a final rigorous proof is still missing. As mentioned above, there is a proof available for the renormalisability of the projectable version of the theory (Barvinsky et al., 2016) but a proof for the complete theory is not available as yet. Such a proof is a fundamental step, since Hořava theory presents itself as a renormalisable UV complete theory of gravity.

The conundrum of coupling matter to a Lorentz violating theory is also a quite important one, as we discussed before. Solutions in this direction do exist as we discussed, but in general there is a naturalness problem arising from the vector sector of the theory. Some mechanisms that allow one to avoid this issue have been proposed, but not much is known yet on how to finally solve the problem. Towards solving the naturalness issues of the theory, particularly in light of the problems that we showed to still be present in the mixed derivative extension of Hořava gravity, we are presented with few different possibilities. One such possibility is that of retaining the mixed derivatives extension by finding a mechanism to eliminate the unstable scalar from the theory. The other possibility is that of abandoning mixed derivatives altogether, and devising a different mechanism to protect the vector sector of Hořava gravity from the quadratic instabilities that generically arise. As to now, a credible solution has not been found and more effort is required to finally solve the conundrum.

The last topic, related to black holes, is one that also needs a lot more investigation in the future. With the results we discussed in this thesis, we laid down the formalism and the background for an understanding, deeper than ever, of black holes and horizons in spacetimes with a preferred foliation. On this turf, many investigations can be started in different directions. In the first place, armed with the formalism we provided in Chapters 4 and 5, numerical simulations of gravitational collapse in the fashion described in Chapter 6 should be performed, both in the simpler setting of spherical symmetry and in more generic settings. There might be the need to expand on the results laid down in Chapter 6 but in the next few years we might finally have a consistent understanding of gravitational collapse and of the emergence of universal horizons in Lorentz violating theories.

From a more practical point of view, black holes are among the most important laboratories at our disposal to test Lorentz violations in gravity. For this reason, it is extremely important to understand known phenomena related to black holes in light of the new theories we are studying; this is the only way we will be able to decide once and for all the fate of Lorentz violating theories, and whether they are compatible or not with what we see in nature. In particular, as a possible direction, the detection of the first binary black hole merger thorugh the gravitational wave signature (LIGO

Collaboration, 2016a,b) offers an invaluable possibility: if we were to have templates for the gravitational wave emission from various objects in the theories we are investigating, we would be able to compare such templates with the observations and be able to confirm the viability of the theories in object.

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